Second quantization:

Recall from quantum mechanics:

\[ [x, p] = xp - px = i\hbar \]

Based on this, let's introduce two operators, which create and annihilate an excitation when applied to the ground state or vacuum.

\[
\begin{align*}
    b &= \frac{1}{\sqrt{2\hbar M\omega}} (Mwx + ip) \\
    b^+ &= \frac{1}{\sqrt{2\hbar M\omega}} (Mwx - ip)
\end{align*}
\]

Let's verify that anticommutator is:

\[
[b, b^+] = 1
\]

\[
[bb^+] = \frac{1}{2\hbar M\omega} \left\{ (M\omega^2) [xx] + iM\omega [px] - \\
    -iM\omega [xp] \right\} = \frac{1}{2\hbar} iM\omega [px]
\]

\[
= \frac{1}{2\hbar} (-2i [xp]) = \frac{-2i}{2\hbar} = 1
\]

Inversely, we can express \( x \) and \( p \) as:

\[
    x = \sqrt{\frac{\hbar}{2M\omega}} (b^++b) \\
    p = i\sqrt{2\hbar M\omega} \left( b^+-b \right)
\]

Recall that \( x, p \) are operators here.

Just a different representation.

For the harmonic oscillator,

\[
    H = \frac{p^2}{2M} + \frac{\beta x^2}{2} = -\frac{k\hbar M\omega}{2\sqrt{2M}} (b^+-b)^2 \\
    + \frac{\beta\hbar}{2M\omega} (b^++b)^2 = \frac{\hbar}{4} (b^+-b)^2 + \frac{\beta\hbar}{2M\omega} (b^++b)^2
\]

\[
\Rightarrow b = \omega x, M
\]
\[ H = \frac{1}{2} \sum \omega (b b^+ + b^+ b) = \frac{1}{2} \sum \omega (n + \frac{1}{2}) \]

where \( n = b^+ b \) is the occupation number (the number of particles in the ground state).

for the states with excitations present:

- \( b^+ |n\rangle = \sqrt{n} |n+1\rangle \) (the philosophy of bosons: the more the merrier)
- \( b |n\rangle = \sqrt{n} |n-1\rangle \)

the amplitude grows as \( \sim \sqrt{n+1} \)

- \( b \) - annihilation
- \( b^+ \) - creation operators
- they increase (decrease) the # of bosons by \( + \) \((-1)\).

for \( n = b^+ b \) \( \Rightarrow b^+ b^+ |n\rangle = b^+ b^+ \sqrt{n} |n-1\rangle = \sqrt{n} \cdot \sqrt{n} |n\rangle = n |n\rangle \)

for the ground state: \( b |0\rangle = 0 \)

\( |n\rangle = \frac{1}{\sqrt{n!}} (b^+)^n |0\rangle \)

Now we are ready for something interesting.

Quantum melting.
Recall that vibration amplitude

\[ u = \delta x = \sqrt{\frac{\hbar}{2\mu w}} (b^+ + b) \] for 1 phonon mode.

The average shift from eq. is \( \langle u \rangle = 0 \) but for:

\[ \langle u^2 \rangle = \frac{\hbar}{2\mu w} \langle (b + b)^2 \rangle = \frac{\hbar}{2\mu w} \langle b^+ b + b b^+ \rangle = \frac{\hbar}{2\mu w} (2 \langle b^+ b \rangle + 1) = \frac{\hbar}{2\mu w} (n + \frac{1}{2}) \]

\[ \frac{\hbar}{2\mu w} b b^+ + b^+ b \]

and for the highest energy phonons

\[ w = w_D, \quad \theta_0 = \frac{2\mu w}{\hbar} \]

\[ \langle u^2 \rangle = \frac{\hbar}{2\mu \theta_0^2} (n + \frac{1}{2}) \]

So for

\[ T \to 0 \quad \langle u^2 \rangle \to \frac{\hbar^2}{2\mu \theta_0^2} \]

(Zero point oscillations)

\[ T \gg \theta_0 : \quad \langle u^2 \rangle \sim \frac{\hbar^2}{\mu \theta_0^2} \]

According to the Lindemann criterion when \( \langle u^2 \rangle \sim a \) melting occurs.

(\( \text{precisely} \ 2\langle u^2 \rangle \sim 0.2a \))

So one can introduce the melting parameter

\[ \Lambda \equiv \frac{\langle u^2 \rangle}{a} < \text{De Boer parameter} \]

at \( T = 0 \quad \langle u^2 \rangle \sim \frac{\hbar^2}{2\mu w} \) where \( w = \sqrt{\frac{\mu}{M}} \) bulk modulus or the stiffness of the lattice

Let's estimate \( \Lambda \)
If we shift an atom from the equilibrium by a there will be a change in the pot.

\[ \text{energ} \sim \text{Ba}^2 \sim V(a) \]

the interaction potential between atoms.

\[ q \sim \frac{V(a)}{a^2} \Rightarrow \Lambda = \frac{h}{q} \sqrt{\frac{M\text{e}^2}{a}} \]

and if \( \Lambda \approx 0.2 - 0.3 \) the system is unstable against the zero point motion and the crystal breaks!!!

Q: Does it really happen?

Well we need small \( m \) and weak \( V(a) \)

meet He. He is very light and the interaction is very small.

That's why He is liquid down to \( -100 \)!

What do you think about \( H \) (hydrogen)?

Another way to "encourage" quantum melting is to apply pressure \( P \)

but of course \( a \downarrow \) but \( V(a) \) goes up and then \( \Lambda \sim V(a) a^2 \) may change either way. For \( He \) \( V(a) \) goes up quickly and it becomes solid. For electronic liquid it is opposite:

\[ V(a) = \frac{e^2}{a} \text{ and } \Lambda = \frac{h}{a} \sqrt{\frac{M\text{e}^2}{a}} \sim \frac{1}{\sqrt{a}} \]

so as \( P \uparrow \) \( a \downarrow \) \( \Lambda \uparrow \Rightarrow \) cold melting of an electronic crystal (Wigner)
Phonons. The 1st Face on collective excitations

Consider a chain of identical atoms:

\[ x = x_0 \]  \rightarrow \text{motion along the chain}

Define the position of atoms as:

\[ x_n \rightarrow \text{in the equilibrium} \quad x_{eq} = n a \]

If we allow the motion of the atoms:

\[ \delta x_n = x_n - x_{eq} \]

Note since it's 1D we allow only longitudinal motion.

Let us also assume the potential is quadratic.

Now it's getting strange. We technically describe the motion of balls connected with springs!!

\[ V_{total} = \sum V(x_i - x_{i-1}) = \]

\[ = N \frac{k}{2} (\delta x_{j+1} - \delta x_j)^2 \]

The force on the mass \( n \) is

\[ F_n = \frac{\partial V_{total}}{\partial x_n} = -\frac{3}{2} \frac{k}{\delta x_n} \left( \frac{k}{2} \left( x_{n+1} - x_n - n \right) \right) \]

\[ \delta x_n = x_n - x_{eq} \]

\[ = k (\delta x_{n+1} - \delta x_n) + k (\delta x_{n-1} - \delta x_n) = m \cdot \delta x_n \]

Recap: A normal mode is a collective oscillation where all particles move at the same frequency.
Let's attempt to solve it by assuming that we have a wave propagating through the chain
\[ \delta x_n = A \cos(kx_n - \omega t) \]
where \( \omega t - ikx_n \) is the phase of the wave.\( \\]
A is the amplitude of the wave.

\[ -m \omega^2 A \cos(kx_n - \omega t) = 2 \frac{e^{-ik^2x_n}}{k} + 2 \frac{e^{ik^2x_n}}{k} \]

or
\[ \omega^2 = 2 \frac{k}{m} \left( 1 - \cos(kx) \right) = 4k \sin^2 \left( \frac{kx}{2} \right) \]

or
\[ \omega = 2 \frac{k}{m} \left| \sin \left( \frac{kx}{2} \right) \right| \]

Not too surprising again, b/c of the periodicity of the crystal we end up with the same physics.

\[ e^{iG \cdot x} = 1 \quad \text{where} \quad G = \frac{2\pi}{a} \cdot n \]

Again the same issue as for electrons. \( k \) is defined up to \( k + 2\pi = k \).

But what \( k \) we should use? say \( \text{say in } \]

\[ \text{phase } = \frac{\omega}{k} \Rightarrow \frac{\omega}{k} = \frac{\omega}{k+G} ? \]
or do we say \[ \frac{2\pi}{m} \geq \frac{2\pi}{k+G} ? \]
The resolution to this puzzle comes from the fact that:

We can define the wave only at the lattice points $x_n = n \cdot a$

So $k + \frac{2 \pi}{a} n$ and $k$ are the same as long as we look only at $x_n$!

This is also known as aliasing of waves.

Properties of 1D chains.

Sound wave: always has $\lambda$ much larger than $a$ (e.g., $h=4$ (cm))

Large $\lambda$, small $k$ so the dispersion $w = \sqrt{\mu k}$

$$w = 2 \sqrt{\frac{k}{m}} / \sin (\frac{ka}{2}) \approx 2 \sqrt{\frac{k}{m}} \cdot \frac{ka}{k}$$

$$\mu_{\text{sound}} = \frac{w}{k} \approx \sqrt{\frac{k}{m} \cdot a}$$

Large $\lambda$, short $k$: We define the group velocity:

\[
\begin{align*}
\frac{d\omega}{d\kappa} & = \frac{d\omega}{d\kappa} \\
\omega_{\text{group}} & = \frac{d\omega}{d\kappa} \\
\omega_{\text{phase}} & = \frac{\omega}{k} \\
\omega_{\text{group}} & = \frac{\omega}{k}
\end{align*}
\]

how individual maxima and minima move.

if the dispersion is linear $\omega_{\text{group}} = \omega_{\text{phase}}$

non-linear they are different.

Note at $\pi/a$: $\omega_{\text{group}} = 0!$ the dispersion is flat.
Two atoms per unit cell. - Phonon modes.

Let's for simplicity assume that \( m_1 = m_2 \).

We are about to repeat the same algebra as for the 1D chain with 1-type of atoms where we found that the system will have the energy dispersion \( \omega(k) \approx \frac{\hbar^2 k^2}{2m} \),

\[
\begin{align*}
\dot{x}_n &= \frac{m}{\hbar} (\ddot{y}_n - \dot{y}_n) + \frac{k_1}{\hbar}(\ddot{y}_{n-1} - \dot{y}_n) \\
\dot{y}_n &= \frac{m}{\hbar} (\ddot{x}_n - \dot{x}_n) + \frac{k_2}{\hbar}(\ddot{y}_n - \dot{y}_{n+1})
\end{align*}
\]

Again we will be looking for a solution of this kind:

\[
\begin{align*}
\delta x_n &= A_x e^{i(wt - i\kappa x_0)} \\
\delta y_n &= A_y e^{i(wt - i\kappa y_0)}
\end{align*}
\]

We focus our attention on the 1st BZ. 

\[-\frac{\pi}{a} < \kappa < \frac{\pi}{a}\), with \( N \) unit cells. 

\( \kappa \) will be quantized

\[
\kappa = \frac{2\pi}{N \alpha} = \frac{2\pi}{N}
\]

Here we mean \( N \) the number of U.C. 
Not the number of atoms.
If we divide the range of \( k \) values in the 1st Brillouin zone by the spacing between neighboring \( k \) values:

\[
\frac{2\pi}{a} = \frac{2\pi}{N_2} \Rightarrow N
\]

we obtain \( N \) possible values of \( k \) or 1 value of \( k \) per unit cell.

But Debye states that there is only one more normal mode per degree of freedom and hence we have 2 atoms but seemingly one mode!

What to do? Let's go ahead and solve the equations:

\[
\begin{align*}
- \omega^2 m A_x e^{-i\omega t - i\xi n a} &= A_x e^{-i\xi n a} + k_1 A_y e^{-i\xi n a} + k_2 A_x e^{-i\xi n a} + (k_1 + k_2) A_x e^{-i\xi n a} \\
- \omega^2 m A_y e^{-i\omega t - i\xi n a} &= k_2 A_y e^{-i\xi n a} + k_1 A_y e^{-i\xi n a} + k_2 A_x e^{-i\xi n a} + (k_1 + k_2) A_y e^{-i\xi n a}
\end{align*}
\]

or

\[
\begin{pmatrix}
\omega^2 m (A_x) \\
\omega^2 m (A_y)
\end{pmatrix} =
\begin{pmatrix}
(k_1 + k_2) & -k_2 - k_1 e^{-i\xi a} \\
-k_2 - k_1 e^{-i\xi a} & (k_1 + k_2)
\end{pmatrix}
\begin{pmatrix}
A_x \\
A_y
\end{pmatrix}
\]
The solution is:

\[
0 = \begin{vmatrix}
(k_1 + k_2) - 2m & -k_2 + k_1 \ e^{ika} \\
-k_2 + k_1 \ e^{-ika} & (k_1 + k_2) - 2m \end{vmatrix} = 0
\]

\[
= \frac{(k_1 + k_2) - 2m}{2} - \frac{(k_2 + k_1) \ e^{ika}}{2} = 0
\]

\[
m \omega^2 = (k_1 + k_2) \pm \frac{1}{2} (k_2 + k_1) \ e^{ika}
\]

\[
\frac{k_2 + k_1 \ e^{ika}}{= \sqrt{(k_1 + k_2) \ e^{ika} - k_1 + k_2 \ e^{-ika}}} =
\]

\[
= \sqrt{k_1^2 + k_2^2 + 2k_1k_2 \cos(ka)}
\]

So finally:

\[
\omega = \sqrt{\frac{k_1 + k_2}{m} \pm \frac{1}{m} \sqrt{k_1^2 + k_2^2 + 2k_1k_2 \cos(ka)}}
\]

for each \( k \) we have 2 solutions \( \pm \)
or 2 branches of the dispersion

or we have 2N modes! Exactly as Debye predicted
one excitation mode per degree of freedom.
Long wavelength show linear dispersion,

\[ \omega = \omega_0 \pm \frac{\hbar k}{m} \pm \frac{\sqrt{1 + \frac{\hbar^2 k^2}{m^2}}}{\hbar^2 k^2} \]

Def: any mode which \( \omega = \pm \hbar k \) for \( k \to 0 \) is called acoustic
and if we assume \( \frac{\tan \frac{\hbar k a}{2}}{\frac{\hbar k a}{2}} \approx \frac{\hbar k a}{2} \) and \( \sqrt{1 + \frac{\hbar^2 k^2}{m^2}} \approx 1 + \frac{\hbar^2 k^2}{2m^2} \)

we get

\[ \omega_0 = \sqrt{\frac{\hbar^2 k_1 k_2}{m^2}} - \frac{\hbar^2 k_1 k_2}{m^2} \frac{(ka)^2}{4} \]

\[ \frac{d\omega}{dk} = \sqrt{\frac{\hbar^2 k_1 k_2}{2m^2 (k_1^2 - k_2^2)}} \]

The high energy branch is more interesting:

For \( k = 0 \)

\[ \omega_+ = \sqrt{\frac{2(k_1 + k_2)}{m^2}} \]

and

\[ \frac{d\omega}{dk} = 0 \mid_{k = 0} \]

This branch is called optical.

Here is why.

During the light absorption, energy and momentum must be conserved, i.e.,

\( \omega = \omega_0 \) so \( \omega \) is very large.

But phonons have \( k \approx \frac{\hbar}{m} \) as maximum and cannot absorb the photon with anything rather than this small value.

But for \( \omega^+ \) even for \( k \to 0 \) \( \omega^+ \) is finite and can match \( \omega = \omega_0 \).

So whenever light + phonons interact we involve optical phonons.
L5

Let's look a little more carefully at the opt. and acoustic mode for $k \to 0$

\[
\begin{pmatrix}
  \omega^2 (A_x) \\
  \omega^2 (A_y)
\end{pmatrix} =
\begin{pmatrix}
  k_1 + k_2 & -k_2 - k_1 \\
  -k_2 - k_1 & k_1 + k_2
\end{pmatrix}
\begin{pmatrix}
  A_x \\
  A_y
\end{pmatrix}
\]

or

\[
\omega^2 (A_x) = \begin{pmatrix}
  1 & -1 \\
  -1 & 1
\end{pmatrix}
\begin{pmatrix}
  A_x \\
  A_y
\end{pmatrix}, \quad \frac{k_1 + k_2}{m}
\]

For $k$ acoustic mode $\omega = 0$ at $k = 0$

\[
\begin{pmatrix}
  1 & -1 \\
  -1 & 1
\end{pmatrix}
\begin{pmatrix}
  A_x \\
  A_y
\end{pmatrix} = 0 \Rightarrow
\begin{pmatrix}
  A_x \\
  A_y
\end{pmatrix} = c
\begin{pmatrix}
  1 \\
  1
\end{pmatrix}
\]

Masses at position $x$ and $y$ move together.

In short the atoms within the unit cell always move together.

For optical branch $k = 0 \quad \omega^2 (k = 0) = \frac{2(k_1 + k_2)}{m}$

\[
\frac{2(k_1 + k_2)}{m}
\begin{pmatrix}
  A_x \\
  A_y
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  -1 & 1
\end{pmatrix}
\begin{pmatrix}
  A_x \\
  A_y
\end{pmatrix}, \quad \frac{k_1 + k_2}{m} \Rightarrow
\]

\[
\begin{pmatrix}
  A_x \\
  A_y
\end{pmatrix} = \begin{pmatrix}
  1 \\
  -1
\end{pmatrix}
\]

describes two modes moving in opposite directions within the unit cell.
Now, let's study our modes close to the zone boundary, i.e., $\pm \frac{\pi}{a}$.

\[ \omega_+ = \sqrt{\frac{2e^2}{mL}} \]
\[ \omega_- = \sqrt{\frac{2m_e}{mL}} \quad \frac{m_e}{m} = 0 \quad \text{at the Z.B.} \]

When the phonon \( k \) approaches the Z. B., the plane wave turns into a standing wave, with anti-phase motion.
For \( n = 0 \) we get \( \frac{\hbar w (k)}{2} \)

The lowest energy of excitation for \( \frac{\hbar w (k)}{2} \) is \( n = 1 \)

\[
\hbar w \overset{\text{for a given } k}{\rightarrow} \frac{\hbar w (k)}{2}
\]

Phonon = a discrete quantum of vibration (photon is a quantum of light)

All known excitations are bosons except for exotics like spinons, monopoles, etc.

Temperature shuffles bosons and the phonons can occupy higher and higher energy states.

Based on the Bose statistics:

\[
\eta_B (\beta \hbar w) = \frac{1}{e^{\beta \hbar w} - 1}, \quad \beta = \frac{1}{k_B T}
\]

Thus the energy expectation:

\[
E_k = \hbar w (k) \left( n_B + \frac{1}{2} \right)
\]

For this we can calculate many thermodynamical quantities, including the heat capacity:

\[
U_{\text{total}} = \sum_k \frac{\hbar w (k)}{2} \left( n_B + \frac{1}{2} \right)
\]

\( k = \frac{2\pi}{N} \cdot m \) for \(-\frac{\pi}{a} \leq k \leq \frac{\pi}{a} \)

As we did for electrons:

\[
\frac{\pi}{2} \rightarrow \frac{\pi}{2(a^2)} \int_{-\pi/a}^{\pi/a} dk
\]

Note the total number of modes is

\[
\frac{Na}{2\pi} \int_{-\pi/a}^{\pi/a} dk = N
\]
Now let's calculate the total internal energy due to phonons:

\[ U_{\text{total}} = \frac{N_s}{2\pi} \int_{-\pi/a}^{\pi/a} \, dk \, \omega(k) \left( N_B \left( \frac{\beta \omega(k)}{k_B T} \right) + \frac{1}{2} \right) \]

\[ \Rightarrow \frac{dU}{d\Gamma} = C_V. \]

Historically, the difference between Debye and what we got is in \( \omega(k) \), and what we got is \( \omega = \frac{\sqrt{6}}{2} \sin \left( \frac{k a}{2} \right) \), but in our case \( \omega(k) \approx \frac{\sqrt{6}}{2} \sin \left( \frac{k a}{2} \right) \)

In fact, what Einstein did in his just introduced the same idea stating how \( \omega \) depends on \( k \).

\[ \text{with } \omega(k) = \sqrt{\frac{3}{\pi}} \frac{c}{a} \sin \left( \frac{k a}{2} \right) \]

\[ \text{with } \omega(k) = \sqrt{\frac{3}{\pi}} \frac{c}{a} \sin \left( \frac{k a}{2} \right) \]

Method of \( \text{Heat Capacity} \):

\[ C_V = \left( \frac{\hbar^2}{2m} \right) \left( \frac{1}{\hbar^2} \right) \left( \frac{1}{\pi^2} \right) \left( \frac{1}{a^2} \right) \left( 1 \right) \left( 1 \right) \left( 1 \right) \]

\[ \text{for a } \text{solid} \]

\[ C_V \text{ statistical states per site is an upper bound of heat capacity} \]

\[ C_V \text{ for a } \text{solid} \]

\[ C_V = \frac{N}{V} k_B T \log \frac{V}{V_0} \]

\[ \text{for a } \text{fluid} \]

\[ C_V \text{ statistical states per site is an upper bound of heat capacity} \]

\[ C_V \text{ for a } \text{fluid} \]

\[ C_V = \frac{N}{V} k_B T \log \frac{V}{V_0} \]
We state without proof that bosons should obey the Bose-Einstein statistics, i.e.
\[ n = \frac{1}{e^{\beta E} - 1} \] 
and if we consider the thermal vibrations of the frequency \( \omega \) (or \( E = \hbar \omega \))
\[ n(\omega; T) = \frac{1}{e^{\frac{\hbar \omega}{k_B T}} - 1} \]
for high temperatures \( k_B T > \frac{\hbar \omega}{2} \)
\[ \frac{1}{2} + h = \frac{1}{2} + \frac{1}{e^{\frac{\beta E}{k_B T}} - 1} = \frac{e^{\frac{\beta E}{k_B T}} + 1}{2 (e^{\frac{\beta E}{k_B T}} - 1)} = \frac{2 + \beta E + \ldots}{\beta E (2 + \beta E + \ldots)} \]
\[ \approx \frac{1}{\beta E} = \frac{k_B T}{\hbar \omega} \quad \Rightarrow \quad \text{for high } T \text{ we have} \]
\[ E = \hbar \omega \left( \frac{1}{2} + n(\omega; T) \right) = \hbar \omega \left( \frac{1}{2} + \frac{k_B T}{\hbar \omega} \right) = \frac{\hbar \omega}{2} + k_B T \]
\( k_B T \) for motion
for 3D = 3 deg. of freedom
\[ E = 3 k_B T \] which is equivalent to the internal energy of a gas.
Density of states for phonons

$\text{U} = \int \text{U}(\mathbf{r},\mathbf{v}) \, d\mathbf{v}$

where $\text{U}(\mathbf{r},\mathbf{v})$ is the number of vibrations between $\mathbf{r}$ and $\mathbf{v}$.

So in the first Brillouin zone of $\mathbb{R}^3$, the number of k-vectors per unit volume is $\frac{8\pi}{3}$.

The volume per unit of the Brillouin zone is $\frac{8\pi}{3} a^3$.

For the high-T limit $C_v = 3R$ per mole (for $\mathbb{R}^3$)

$C_v = \frac{\partial^2 E}{\partial T^2}$

For the classical limit $\text{C}_{\text{v, class}}$ cannot explain why the heat capacity $\to 0$ as $T\to 0$.

Dulong-Petit law $C_v = \frac{3}{2} R$.

Heat capacity follows $C_v = \frac{3}{2} R$ for $\mathbb{R}^3$. No additional explanation is needed.
or in spherical coordinates:

\[ g(\omega) \, d\omega = \frac{V}{(2\pi)^3} \cdot \frac{\omega}{V_n} d\omega \]

\[ = \frac{h}{2\pi^2 V_n} \cdot \frac{d\omega}{\omega} \]

so \( g(\omega) \sim \omega^2 \)

**Heat Capacity**

The total energy due to phonons:

\[ E = \int_{0}^{\omega_0} \frac{3V\omega^2}{2\pi^2 V_n} \cdot \frac{1}{e^{\frac{\hbar \omega}{k_B T}} - 1} \, d\omega \]

where \( \omega_0 \) is called the Debye frequency.  

\[ \Rightarrow \frac{dE}{dT} = CV = \int_{0}^{\omega_0} \frac{3V\omega^2}{2\pi^2 V_n} \cdot \frac{\hbar \omega}{k_B T} \, d\omega \]

**Low T limit:** At low T only low \( \omega \) phonons exist since \( n(\omega, T) \rightarrow_0 T \rightarrow 0 \) 

also those \( \omega \ll \omega_0 \) so \( \omega_0 \) looks like \( \omega_0 \).

Substituting \( x = \frac{\hbar \omega}{k_B T} \)

\[ E = \frac{3V\hbar^2}{2\pi^2 V_n} \left( \frac{k_B T}{\hbar} \right)^3 \int_{0}^{\infty} x^3 \left( e^x - 1 \right)^{-1} \, dx \]

\[ \Rightarrow E = \frac{V\hbar^2 (k_B T)^3}{10 (\hbar v_n)^3} \]

\[ \Rightarrow \frac{dE}{dT} = CV = \frac{\frac{dE}{dT}}{T^3} \]

or \( CV = \frac{2V\hbar^2 k_B}{3\left(\frac{\hbar v_n}{k_B T} \right)^3} \cdot T^3 \)

if we introduce the Debye temperature

\[ \Theta_D = \frac{\hbar \omega_0}{k_B} \Rightarrow \Theta_D = \frac{\hbar \omega_0}{k_B} \left( \frac{16\pi N^2}{V} \right)^{1/2} \]

(not shown here and APM)