INTEGEGER

HALL

EFFECT

(integer) Quantum Hall Effect.

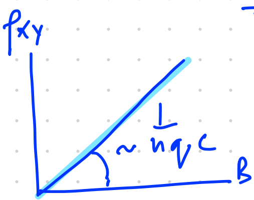
Lets place our material into a <u>high</u> mag. field

Recall from the lecture on T we derived that

$$j_\mu = \sigma_{\mu\nu} E_\nu \rightarrow$$

$$\rightarrow E_\mu = \rho_{\mu\nu} j_\nu$$

or for the case of $B = B\hat{z}$ we have



$$\rho_{xy} = \frac{B}{nqc}$$

ρ_{xy} v.s. B can be used to measure carrier density n and the sign of carriers, q.

Now let's place a 2D electron gas and do the same measurement.

This looks completely different!!!

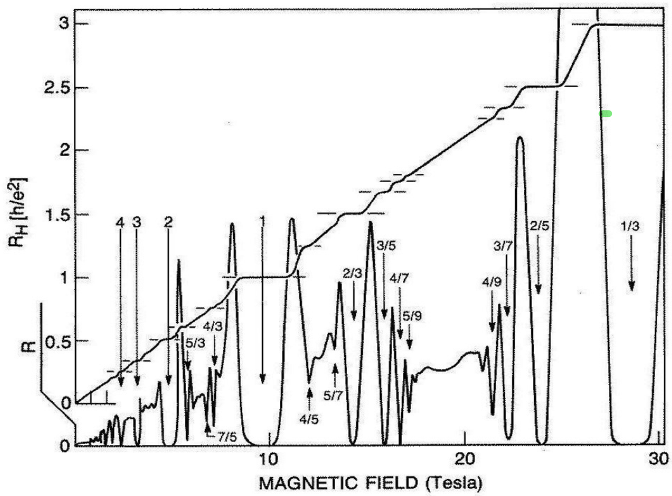


Figure 5: Plots of Hall and longitudinal resistivities as a function of magnetic field. Plateaus in Hall resistivity at several fillings are visible. After Ref. [28]

First we see a set of plateaux in which

$R_H = \frac{V_y}{I_x} = \rho_{xy}$ is quantized in units of

$$\frac{e^2}{h} \qquad = \quad \underline{\nu \, \frac{h}{e^2}}$$

(1980 IQHE von Klitzing, Dorda Pepper)

This feature is universal, independent of material, impurities etc.

But according to the plateaux $\rho_{xx} \to 0$ (dissipation less!) : drops by 13 orders.

in 1982, Tsui, Störmer and Gossard showed that in the presence of disorder ν becomes fractional = FQHE which involves strong electronic correlations.

Fermions condens into quantum states with excitations carrying fractional quantum numbers

$$\Downarrow$$

Topologically ordered phases.

$\frac{h}{e^2}$ = quantum units of resistance.

$$n = \frac{N}{A}$$

$$R_H = \frac{B}{nec} = \frac{h}{e^2} \frac{Be}{nhc} = \frac{h}{e^2} \frac{B \cdot A}{hc/e} \cdot \frac{1}{nA}$$

$$= \frac{h}{e^2} \cdot \frac{\Phi}{\Phi_0} \cdot \frac{1}{N_e} = \frac{h}{e^2} \cdot \frac{N_\Phi}{N_e},$$

Here A is the area of the sample,

$N_e \equiv nA$ # of e^-s

$\phi = BA$ flux through the system.

$\phi_0 = \frac{hc}{e}$ flux quantum

N_ϕ = # of the flux quanta =

$\frac{N_e}{N_\phi}$ $\equiv$ a Landau-level filling factor.

Why do we care about 2D?

R = Resistivity for the d-dimensional system

$= \rho L^{(2-d)}$ if $d=2$ $\boxed{R \equiv \rho}$ in 2D only

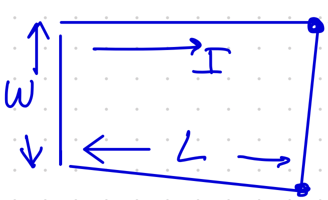
Resistance

Meaning that this quantity is a scale invariant!

in 2D $[R] = [\rho] = \left[\frac{h}{e^2}\right]$ the same units

But R is still geometry dependent

$$\boxed{R = \rho \frac{L}{W}}$$



$$R_{xy} = \frac{V_y}{I_x}$$

Notice $\frac{L}{W}$ describes geometry (SHAPE) and not it's size.

For Hall $= R_{yx} = \frac{V_y}{I_x} = \frac{E_y W}{I_x} = E_y/(I_x/W)$

$= \frac{E_y}{J_x} = \rho_{yx} \leftarrow$ no $\frac{W}{L}$ at all!

Moreover since it's dissipationless transport the location of ~~contacts~~ contacts is unimportant.

As there is no voltage drop it's isopotential

Notice, the lack of dissipation is b/c of a quantum effect $\Rightarrow$ quenching of kinetic energy by strong mag. field.

in short Hall resistance = Hall resistivity

Why disorder is important?

Strangly but true the universality of IQHE is due to disorder?!

This is the case of Anderson localization. which in 2D any amount of disorder should cause the localization.

Specifically, in the absence of disorder 2DEG is <u>translationally invariant</u> (ignore the periodic potential of the xtal!)

if we use the frame which moves with $-v$ w.r.t. lab frame

$$\bar{J} = -ne\bar{v}$$

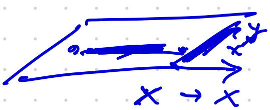
In the Lab frame $\bar{E} = 0$

$\bar{B} = \bar{B}\hat{z}$

in the moving frame: $\bar{E} = -\frac{1}{c}\bar{v}\times\bar{B}$
(Lorentz transformation) $\bar{B} = B\bar{z}$

We need to cancel ~~this~~ the Lorentz force $-\frac{e}{c}\bar{v}\times\hat{B}$ by this electric field

$$\bar{E} = \frac{B}{nec}\bar{J}\times\bar{B}$$



Recall $E_\mu = \rho_{\mu\nu}J_\nu \Rightarrow$

ρ

$$\underset{dis.}{\rho} = \frac{B}{enc}\begin{array}{c} \begin{array}{cc} \rho_{xx} & \rho_{xy} \end{array} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array} \Rightarrow \underline{\underline{\sigma}} = \frac{nec}{B}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\neq$ SC!

since $\sigma_{xx} = 0$ it's an insulator?!

but $\rho_{xx} = 0 \Rightarrow$ perfect conductor?!

(normally if $\sigma_{xy} = 0$ $\rho_{xx} \to \infty$)

But $\boxed{\sigma_{xy} = -\frac{nec}{B} \neq 0}$, so it's not a superconductor.

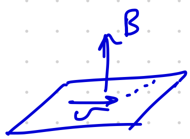
Notice the only info about 2DEG we need is n, if there is no disorder.

So if we had a perfect sample we would not find any new physics?!
(b/c of translational invariance)

Thus we need disorder to kill the tr. invar.

semi-Classical transport:

in 2D:



$$\left\{\begin{aligned} m\ddot{x} &= -\frac{eB}{c}\dot{y} \\ m\ddot{y} &= +\frac{eB}{c}\dot{x} \end{aligned}\right\} \longrightarrow \cancel{\dot{y}}\ \ddot{y} = -\frac{cm}{eB}\dddot{x}$$

$$\boxed{m\left(-\frac{cm}{eB}\right)\dddot{x} = +\frac{eB}{c}\dot{x}}, \quad \text{call } \boxed{\dot{x} = z},$$

$$\boxed{-\frac{m^2c}{eB} z'' = \frac{eB}{c} z} \Rightarrow z'' = -\frac{e^2B^2}{m^2c^2} z$$

$\Downarrow$

oscillator with $\boxed{\omega^2 = \left(\frac{eB}{mc}\right)^2}$

or $$\boxed{\omega_{cyclotron} = \frac{eB}{mc}}$$

and $(x, y) = \bar{r} = R\,(\cos(\omega_c t + \varphi), \sin(\omega_c t + \varphi))$

$R \uparrow$ any const

$T = \frac{1}{\omega_c}$ independent of R!

$v = R\omega_c$

This kind of motion is called isochronous

e.g. in oscillator where T is indep. of amplitude

Let's review the approach to canonical momentum: Classical mech:

$$\mathcal{L} = \frac{1}{2} m \dot{x}^\mu \dot{x}_\mu - \frac{e}{c} \dot{x}^\mu A^\mu$$

$\bar{A}$ = vector potential at x^μ

The eqn. of motion

$$\frac{\partial \mathcal{L}}{\delta x^\nu} = -\frac{e}{c} \dot{x}^\mu \partial_\nu A^\mu$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\nu} = m_e \dot{x}^\nu - \frac{e}{c} A^\nu$$

The Euler - Lagrange eqn. of motion

$$m \ddot{x}^\nu = -\frac{e}{c} \left[\partial_\nu A^\mu - \partial_\mu A^\nu \right] \dot{x}^\mu$$

or by usin $B = \nabla \times A$

$$B^\alpha = \epsilon^{\alpha\beta\gamma} \partial_\beta A^\gamma$$

or $$\boxed{m \ddot{x}^\nu = -\frac{B e}{c} \dot{x}^\mu}$$

Once w have the Lagrangian the canonical momentum

$$\boxed{p^\mu = \frac{\delta \mathcal{L}}{\delta \dot{x}^\mu} = \boxed{\underbrace{m_e \dot{x}_\mu}_{p} - \frac{e}{c} A^\mu}}$$

And the hamiltonian

$$H[p,x] = \dot{x}^\mu p^\mu - \mathcal{L}(\dot{x},x) =$$

$$= \dot{x}^\mu \left(m\dot{x}^\mu - \frac{e}{c}A^\mu\right) -$$

$$- \frac{1}{2}m\dot{x}^\mu \dot{x}^\mu - \frac{e}{c}\dot{x}^\mu A^\mu =$$

$$= \frac{1}{2m}\left(p^\mu + \frac{e}{c}A^\mu\right)\left(p^\mu + \frac{e}{c}A^\mu\right) =$$

$$= \frac{\Pi^\mu \Pi^\mu}{2m}$$

mechanical momentum

Hamilton eqn:

$$\dot{x}^\mu = \frac{\partial H}{\partial p^\mu} = \frac{\Pi^\mu}{m}$$

$$\dot{p}^\mu = -\frac{\partial H}{\partial x^\mu} = -\frac{e}{mc}\left(p^\nu + \frac{e}{c}A^\nu\right)\partial_\mu A^\nu$$

So $\boxed{\Pi^\mu = m\dot{x}^\mu}$

Let's quantize it semiclassically.

To make a big fat ~~e~~ electron we introduce a wave packet made of Bloch waves: $\Psi_{R(t),K(t)}(r,t)$

The packet must large $\gg \lambda_{\text{de Broglie}}$
so we can define a central $K(t)$ and $R(t)$
(no Heisenberg here!)

from semiclassical theory (see Ch. 8 of the main text)

$$\begin{cases} \dot{R}^{\mu} = \dfrac{\partial \langle \Psi_{R,K} | H | \Psi_{R,K} \rangle}{\partial (\hbar K^{\mu})} \\[2ex] \hbar \dot{K}^{\mu} = -\dfrac{\partial \langle \Psi_{R,K} | H | \Psi_{R,K} \rangle}{\partial R^{\mu}} \end{cases}$$

This works for weak mag. field and fast electrons.

e.g. $\hbar \omega_c \ll E_F$

or in the oscillator we have $\hbar\omega_c \sqrt{B}$ $\approx 25\,\text{nm}/\sqrt{B}$

and $\ell = \sqrt{\dfrac{\hbar c}{eB}} \approx \dfrac{257\,\mathring{A}}{\sqrt{B(\text{Tesla})}} = \dfrac{257\,\mathring{A}}{1T} \approx 25\,\text{nm}$

The physical meaning inside the area $2\pi\ell^2$

we have exactly 1 quantum of flux

$\Phi_0 = \frac{hc}{e}$ $\Rightarrow$ so the density of magn. flux

$$B = \frac{\Phi_0}{2\pi \ell^2}$$

And now for the quantum version in the strong field.

1st let's select a gauge for the vector potential, e.g., Landau gauge

$$\bar{A}(\bar{r}) = -xB\bar{y} \Rightarrow \nabla \times \bar{A} = -B\bar{z}$$

use $-$ for future simplification

So the potential points in $\bar{y}$ but varies with x!

Also, the system is translationally invariant in y

so translation leaves $\nabla \times \bar{A}$ invariant in y but not $\bar{A}$ itself.

Next, we ignore the band structure due to the periodic potential, and we can write the Hamiltonian in the Landau gauge as: $H = \frac{1}{2m}\left(p_x^2 + \left(p_y - \frac{eB}{c}x\right)^2\right]$

b/c $\bar{A}$ points in y.

B/c system is translational in y, we separate variables as:

$$\psi_k(x, y) = e^{iky} f_k(x)$$

So for $H\psi_k = \epsilon_k \psi_k$ or

$$\mathcal{H} f_k(x) = \epsilon_k f_k(x)$$

where $\mathcal{H} = \frac{p_x^2}{2m} + \frac{1}{2m}\left(\hbar k - \frac{eB}{c}x\right)^2$

$$= \frac{p_x^2}{2m} + \frac{1}{2} m \omega_c^2 \left(x - k\ell^2\right)^2$$

$$\ell = \sqrt{\frac{\hbar c}{eB}}$$

$\odot$

Let's label $k\ell^2 \equiv X_k$ we end up

$$\left[\frac{p^2}{2m} + \frac{1}{2} m \omega_c^2 (x - X_k)\right] f_k(x) = \epsilon_k f_k(x)$$

this is 1D eqn for a harmonic oscillator with

$$\boxed{\epsilon_{k,n} = (n + 1/2)\hbar\omega_c}$$

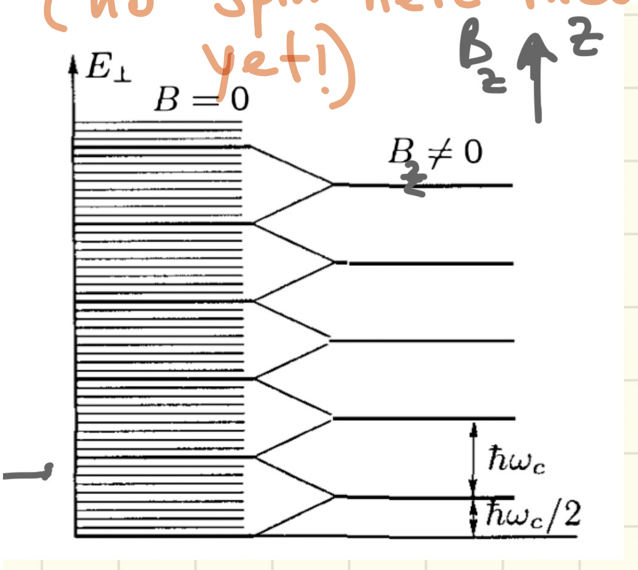
The eigenstate is:

$$\psi_{n,k}(r) = e^{iky} H_n\left(x/\ell - k\ell\right) e^{-\frac{(x-k\ell^2)^2}{2\ell^2}}$$

this can be verified by the direct substitution.

These levels are called Landau levels

(no spin here included yet!)



Energy Spectrum of quasiparticles in Magnetic field

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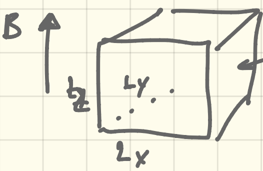
Ideal gas of electrons: (no Spin) yet



box of free electrons

$$E(\bar{p}) = \frac{|p|^2}{2m_0}$$ let's separate those

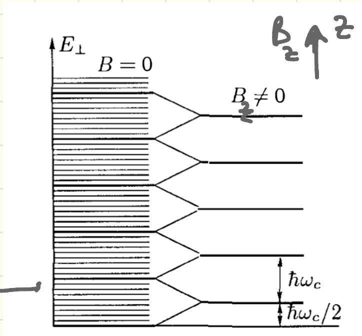
$$E = \frac{p_x^2 + p_y^2}{2m_0} + \frac{p_z^2}{2m_0} = E_\perp + E_\parallel$$

Recall the density of states for 2D is const(E)

$$\nu^{2D}(E) = \frac{m^*}{\pi\hbar^2} \quad \text{(2D only)}$$

- Every energy level is degenerate

for each E we have many p_x and p_y such as $p_x^2 + p_y^2 = 2m_0 E_\perp$ for many n_x and n_y



in the plane $\perp$ to $\bar{B}$ electrons move on the circle of $r_B = \dfrac{m_0 v_\perp}{Be}$ with $\omega_c = \dfrac{eB}{m_0}$

↳ the energy is quantized:

$$E = E_\perp + E_\parallel = \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{p^2}{2m_0}$$

$n = 0, 1, 2 \dots$

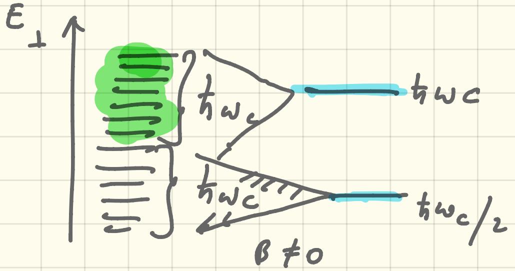
$n_x, n_y = 0, 1, 2, \dots$

For energy $E_\perp$ we have only energies $= \hbar\omega_c(n+1/2)$ separated by $\hbar\omega_c$

For $\parallel$ we get $E_\parallel = \frac{p_z^2}{2m_0} = \frac{\hbar^2}{2m_0}\left(\frac{2\pi\hbar}{L_x}\right)^2 n_z^2$

huge # of states almost quazicontinious.

See page 5



so already degenerate spectrum of $B=0$ now IS VERY degenerate

The # of e^- in the band of size $\hbar\omega_c$

$\nu = \frac{dN}{dE}$

$$N_L = \underbrace{\nu^{2D}(E)}_{\text{density of 2D states}} \cdot \underbrace{\hbar\omega_c}_{\Delta E} = \overbrace{S}^{=L_x \cdot L_y} \frac{m_0}{\pi\hbar^2} \cdot \hbar\omega_c = \frac{L_x L_y m_0 \hbar\omega_c}{\pi\hbar^2}$$

Const for 2D

N_L defines the degree of degeneracy of $E_\perp$ for $B\neq 0$.

For descrete values of $E_n^\perp$ in the quaziclassical approximation corresponds a specific trajectory; which depends on the quantum # n. Then our condition:

$$\lambda_B \ll r_{B_n} \text{ is equal } \hbar\omega_c \ll E_F$$

To find the radius r_{B_n} lets compare

$$E_{\perp}^{\text{classical}} = \frac{m_0 \omega_c^2 r_{B_n}^2}{2} \quad \text{and } E_{\perp}^{\text{quantum}} = \hbar\omega_c(n+\tfrac{1}{2})$$

From this we get

$$\boxed{r_{B_n} = \sqrt{\frac{2\hbar}{m_0 \omega_c}\left(n+\frac{1}{2}\right)} = \sqrt{\frac{2\hbar}{eB}(n+1/2)}}$$

- So for the electron to go from the orbit n to $n+1$ needs to get a nice $\hbar\omega_c$
- For the same n e^- get the same r_{B_n} But n_z and P_z can be different

<u>LETS INCLUDE SPIN</u>

for electron with mag. moment $\mu_B = e\hbar/2m_0c$ its energy in $B = -\bar{\mu}\bar{B}$; with the spin we split a Landau level into 2 sublevels dependin if $\mu \uparrow\uparrow B$ or $\mu \uparrow\downarrow B$

$$E(n, s, k_z) = \hbar\omega_c(n+1/2) + \underbrace{s}_{s=\pm 1}\mu_B B + \frac{\hbar^2 k_z^2}{2m_0}$$

$$\left.\begin{array}{lll} s = +1 & \Rightarrow & \text{"+" state} \\ -1 & \ldots\ldots & \text{"–"} \ldots \end{array}\right\} \text{the lowest level } 0^-$$

Note: Spin removes the Landau degeneracy for the same n we have $n, s=+1$ $n; s=-1$

(see page 8)

This note should be on the previous page:

The distance between LLs is very small

i.e. $\hbar\omega_c = 0.18\ (meV/T) \cdot B \frac{m_e}{m^*}$

so e.g. if $m^* = m_e$ and $B = 20T$

we get $\hbar\omega_c \simeq 0.36$ meV !

Extremely small # we cannot measure experimentally, e.g. ARPES > 1-2 meV even at 10K.

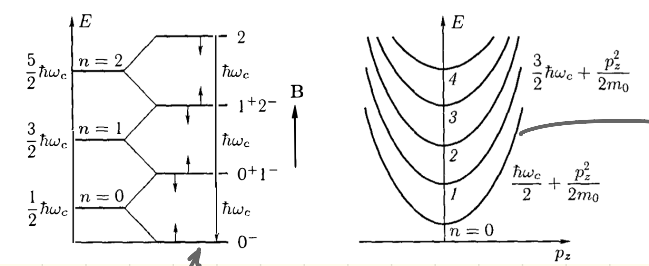
Thus we must:

- reduce disorder to the absolute min
- have a system with very light e^-

e.g. m^* GaAs = 0.067 meV

- in metals we cannot observe the QHE b/c life-time of e^- < time for a full cyclotron orbit.

No orbit no quantization.

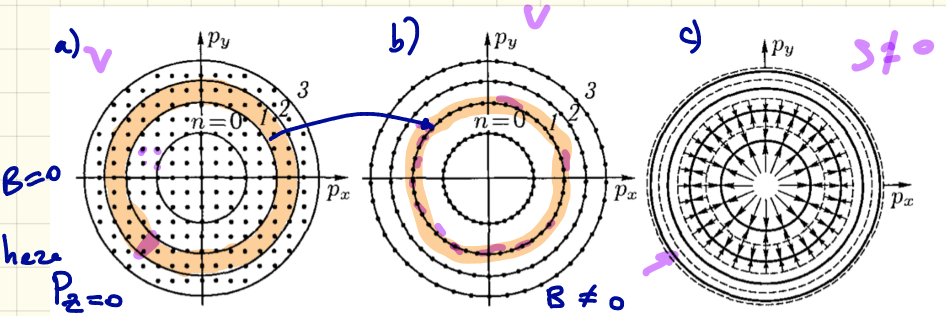


each state on the parabola is strongly degenerate

note spin degeneracy is only absent for 0^-.

Since E continiously depends only on p_z it looks like we have a quasi-1D system!

DISTRIBUTION OF ELECTRONS in p-space



Assume for have a single zone metal with a spherical Fermi surface.
if $B=0$ all the states are inside the sphere and occupy $(2\pi\hbar)^3$. So we mark the py points separated by $2\pi\hbar$. The maximum circle is $p_F = \sqrt{2m^* E_F}$. For any other x-section the states fill up the circle of $\sqrt{p_F^2 - p_z^2}$; as $p_z \to p_F$ the radius goes $\to 0$.

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The uniform distribution of states with $p_x, p_y\ p_z$ corresponds to $E = E(p_x\ p_y\ p_z)$ where $0 < |p| < p_F$

Now we turn on B: in the plane $\hbar\omega_c(n + 1/2)$ for $p_z = \text{const}$, to find the radius p_n we write down

$$E_\perp^{\text{classic}} = \frac{p_x^2 + p_y^2}{2m^*} = \frac{p_n^2}{2m^*}$$

$$= E_\perp^{\text{quantum}} = \frac{1}{2}\hbar\omega_c(n + 1/2)$$

$$\Rightarrow \quad p_n = \sqrt{2m\hbar\omega_c(n + 1/2)}$$

(see fig b in page 8)

In other words: all states which we had confined inside the orbits with a radiu p_n $n = 0, 1, 2, \dots$, now collapse on the circles see fig. a vs. b in page 8

Note the area in a) $\pi(p_{n+1}^2 - p_n^2) =$

$$= 2\pi \underbrace{m^*\hbar\omega_c}_{r_{Bn}}$$

Except for 0^- state:

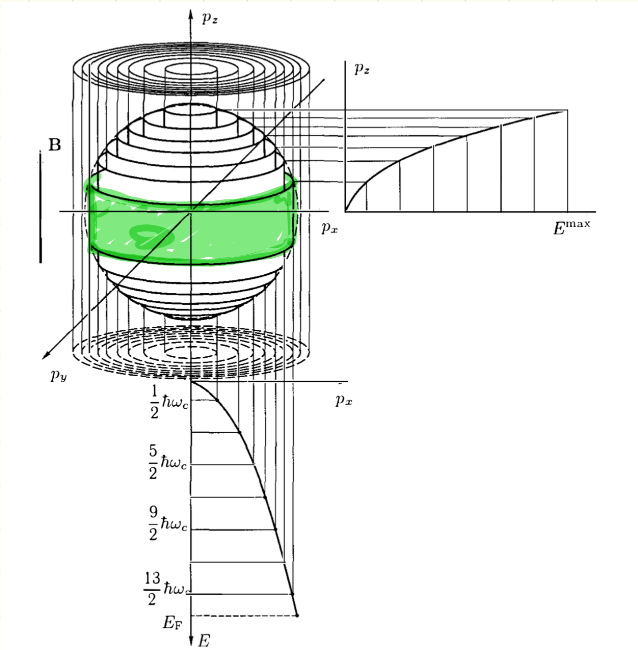
$$\pi p_0^2 = \pi m^* \hbar\omega_c$$

So for each allowed orbit we have the same # of e^- $N_L = \dfrac{m^* L_x L_y \hbar\omega_c}{\pi\hbar^2} \Rightarrow$

degeneracy of those p_n orbits is the same as the descrete Landau levels

Note since p_n is independent of p_z all orbits are of the same radius then we deal with the Landau cylinders

10



- number of states filled up by e^- on each cylinder depends on its length within p_F
- with increasing p_n length ↓
- # of cylinders ↓ with increasing ↑

To be cont'd:

Topological p.o.v. for IQHE

Topological properties of IQHE

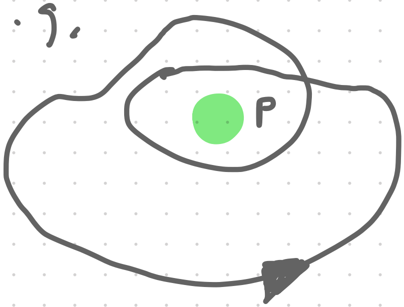
Global
√ Geometrical properties of an object in the mathematical space.

e.g. k-space for the electron in the Hilbert space.

The goal is to classify objects based on geometrical properties:

- bending, stretching ok
- ~~poking holes and glueing~~ is NOT!

e.g.



how many times the loop winds up before it enclose the point P.

"Answer: 2 times".

Lets try this mathematically:

1. 1^{st} we define the function

$$z(t)\ ,\ t \in [0,1]\ ,\ t \in \mathbb{R}\ ,$$
$z \in \mathbb{C}$, as usual

$$z(t) = |z(t)| \cdot e^{i\varphi(t)}$$

Now we can define the integral!

$$\boxed{Q_I(z) = \frac{1}{2\pi i}\int_0^t \frac{dz(t)/dt}{z(t)}\,dt}$$

Lets confirm the Q_I is the quantity we want:

$$Q_I(z) = \frac{1}{2\pi i}\int_0^1 d/dt\left(\ln(z(t))\right)dt =$$

$$= \frac{1}{2\pi i}\ln(z(t))\Big|_0^1 = \frac{1}{2\pi i}\ln\left|\frac{z(1)e^{i\varphi(1)}}{z(0)e^{i\varphi(0)}}\right|$$

$$= \frac{1}{2\pi i}\cdot\left(\ln\left(e^{i\varphi(1)} - \ln e^{i\varphi(0)}\right)\right) =$$

$$= [\varphi(1) - \varphi(0)]/2\pi$$

if $\varphi(t)$ is continious, i.e. no jump from $2\pi \to 0$ after the turn $\Rightarrow$ $Q_I(t)$ gives the number of turns!

Most important we classify all possible paths in 2D:

Obviosly $Q_1(\tau) \in \mathbb{Z}$ $\Rightarrow$ is called $Q_I(\tau)$ is a $\mathbb{Z}$-type topological invariant.

Lets apply this concept to IQHE

For this purpose we rederive Hall conductivity tensor quantum-mechanically in Kubo approximation.

1) Bloch state in a solid is $\psi_{nk}(x) = U_{nk}(x) e^{ikx}$

n - band index

k - wave vector

2) Apply 1^{st} order perturbation theory in the weak electric field $\bar{E} = E_x \cdot \bar{e}_x$

the electric potential $(B=0)$

$$\phi_{el}(x) = eE_x \cdot x = \underbrace{-i\frac{d}{dk_x}}\cdot eE_x$$

The perturbed w.f.

$$|n\rangle = |n_0\rangle - \underbrace{\sum_{m_0 \neq n_0} \frac{|m_0\rangle\langle m_0| ie E_x \cdot \frac{d}{dk_x}|n_0\rangle}{E_{n_0} - E_{m_0}}}$$

E_{n0} E_{m0}

$|n_0\rangle$ and $|m_0\rangle$ solutions of unperturbed hamiltonian:

Lets determine the velocity in y direction

$$v_y = \langle n | v_y | n \rangle = \langle n_0 | v_y | n_0 \rangle - ie E_x \sum_{m_0 \neq n_0} \frac{\langle n_0 | v_y | m_0 \rangle \langle m_0 | \frac{d}{dk_x} | n_0 \rangle}{E_{n_0} - E_{m_0}} + \text{h.c.}$$

② = from pert $\vec{E}$

①

$$v_y = dy/dt = -\frac{i}{\hbar}[\hat{H}, y]$$ ← verify from $i\hbar \frac{\partial \psi}{\partial t} = H\psi$

$$\langle n_0 | v_y | m_0 \rangle = -\frac{i}{\hbar}\left(\langle n_0 | Hy | m_0 \rangle - \langle n_0 | yH | m_0 \rangle\right)$$

$$= -\frac{i}{\hbar} \langle n_0 | y | m_0 \rangle \cdot (E_{n_0} - E_{m_0})$$

$$y = i \frac{\partial}{\partial k_y}$$

①

$$\langle n_0 | v_y | m_0 \rangle = -\frac{i}{\hbar} \langle n_0 | \frac{\partial}{\partial k_y} | m_0 \rangle (E_{n_0} - E_{m_0}) =$$

$$= -\frac{i}{\hbar} \left\langle \frac{d n_0}{d k_y} \middle| m_0 \right\rangle \cdot (E_{n_0} - E_{m_0})$$

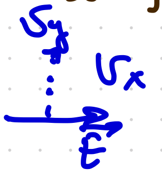
for all $m_0 \neq n_0$. Insert into v_y

$$v_y = \underbrace{\langle n_0 | v_y | n_0 \rangle}_{0} + \frac{ieE_x}{\hbar} \sum_{m_0 \neq n_0} \left\langle \frac{\partial n_0}{\partial k_y} \middle| m_0 \right\rangle \cdot \left\langle m_0 \middle| \frac{\partial n_0}{\partial k_x} \right\rangle + \text{h.c.}$$

$$\sum_{m_0 \neq n_0} |m_0\rangle\langle m_0| = 1$$

from $\langle n_0 | v_y | n_0 \rangle = \frac{i}{\hbar} \langle n_0 | \frac{\partial}{\partial k} | n_0 \rangle \underbrace{(E_{n_0} - E_{n_0})}_{=0} = 0$!

So finaly



$$v_y = \frac{ieE_x}{\hbar}\left(\left\langle \frac{\partial n_0}{\partial k_y}\middle|\frac{\partial n_0}{\partial k_x}\right\rangle - \underbrace{\left\langle \frac{\partial n_0}{\partial k_x}\middle|\frac{\partial n_0}{\partial k_y}\right\rangle}_{h.c.}\right) \neq 0 \text{ maybe}$$

and since the plane wave part

in $\boxed{|n_0\rangle = \underline{u_{nk}(\bar{x})}\, e^{i\bar{k}\bar{x}}}$ doesn't contribute

we finaly get:

$$\boxed{v_y = \frac{ieE_x}{\hbar}\left(\left\langle \frac{\partial u_{nk}(\bar{x})}{\partial k_y}\middle|\frac{\partial u_{nk}(\bar{x})}{\partial k_x}\right\rangle - \left\langle \frac{\partial u_{nk}(\bar{x})}{\partial k_x}\middle|\frac{\partial u_{nk}(\bar{x})}{\partial k_y}\right\rangle\right)}$$

linear response theory based on Kubo formalism.

To get current I_y in the electric field E_x, we add up all the contributions from all occupied states $u_{nk}(\bar{x})$.

$$j_y = -e \cdot v_y$$

$\Rightarrow$ The transverse current $\neq 0$

if $\frac{\partial U}{\partial k_x}$ and $\frac{\partial U}{\partial k_y}$ ARE Different!

and contribution from different k, should not cancel.

Now we need to prove that the same Bloch w.f. works for a magnetic field.

Recall the translation operator ($B=0$)

$$T(R_n) = e^{\vec{R}_n \cdot \nabla}$$

$$T(R_n) \cdot f(\bar{x}) = f(\bar{x} + \vec{R}_n) \Rightarrow$$

$T(R_n)$ commutes with $V(\bar{x})$ $\Rightarrow$ $TV(\bar{x}) = V(\bar{x}+\bar{R}_n) = V(\bar{x})$

it also commutes with ∇^n $n=1,2,\ldots$

$\Rightarrow$ it commutes with $\boxed{H = -\frac{\hbar^2}{2m}\nabla^2 + eV(\bar{x})}$ $B=0$

$\Rightarrow$ eigenstates of H and T are common $\Rightarrow$ exactly Bloch functions

Now we apply an external mag. field

$$\hat H_B = \frac{1}{2m}\left(i\hbar\nabla + e\hat A(x)\right)^2 + e\hat V(x)$$

where $\hat A(x) = -\frac{1}{2}\left(\hat{\bar x} \times \hat{\bar B}\right)$

↑ in symmetric gauge

Since $A(\bar x) \neq A(\bar x + \bar R_n)$ $T(R_n)$ doesn't commute with $\hat H_B$, but

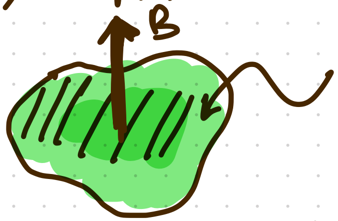
new $$\boxed{T_B(\bar R_n)} = \underbrace{e^{\bar R_n \cdot \left(\nabla - \frac{e}{i\hbar}\hat A(\bar x)\right)}}_{\text{no field}}$$

will commute with $i\hbar\nabla + eA(\hat x)$ ← Show THIS!

But now the problem is:

$$T_B(R_n)\,V(\bar x) = \underbrace{e^{R_n \cdot \frac{e}{i2\hbar}\left(\vec x \times \bar B\right)^{=A(x)}}}_{\text{extra phase}} \cdot V(x + R_n)$$

Imagine we now move in the loop by applying the operator $T_B(R_n)$ many times.



area size $\tilde A$

$$i\frac{eB}{\hbar}\cdot\tilde A = i\frac{2\pi eBA}{\hbar}$$

= if integer

e.g. like in the Aharonov-Bohm

$$\oint A(\bar x)\,d\bar x = \iint \nabla \times A(\bar x)\,d\tilde A = \iint B\,d\hat A = |\bar B|\cdot\tilde A\cdot\operatorname{sgn}(\bar B\cdot\vec{\tilde A})$$

b/c $e^{i\cdot 2\pi} = 1$ the phase will vanish if $\tilde{A}$ contains an even number (!) of magnetic flux quanta ϕ

$$\underline{\underline{2\pi i}} \rightarrow \frac{e\, B\tilde{A}}{\hbar} = n \quad \text{or} \quad \boxed{\frac{\phi}{\phi_0} = \underline{\underline{2n}} = \text{integer even}}$$

$= \phi_0$

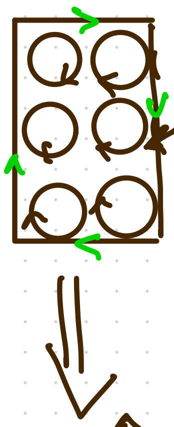
Note the flux quantum is independent of gauge $\bar{A}(\bar{x})$

$\Rightarrow$ We can now define a new unit cell that contains an even number of flux quanta. The new unit cell is called the magnetic U.C. with lattice vectors $\vec{R}_{n,B}$. The Shrodinger equation commuts with $\hat{T}_B(\bar{R}_{nB})$ and hence Bloch w.f. still good for $B \neq 0$.

$\Rightarrow$ But b/c of the extra phase $e^{-\frac{e}{i\hbar}\hat{A}(\bar{x})}$ in $T_B(R_{n,B})$

$u_{n,k}(\bar{x})$ is not simple but $u_{n,k} = |u_{n,k}(\bar{x})| \cdot e^{i\theta_{n,k}(\bar{x})}$

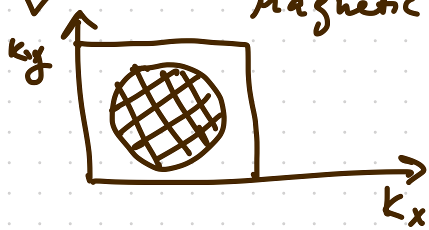
with $\oint_{\text{magn. unit. cell}} \frac{d\theta_{n,k}(\bar{x})}{ds}\, ds = -2\pi p$



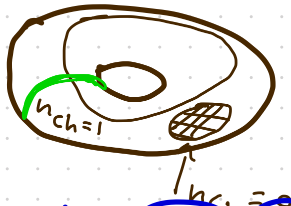
a magnetic unit cell

$p = 3$ pairs

magnetic BZ



$\Rightarrow$ Periodic boundary Condition



$j = -e \cdot v_y$

Finally we calculate

$$\dot{j}_y = -e \oiint_{\text{Mag. BZ}} \frac{1}{(2\pi)^2} \hat{v}_y(\bar{k})\, d\bar{k} =$$

$$= -e \oiint \frac{1}{(2\pi)^2} \frac{ieE_x}{\hbar = h/2\pi} \left(\left\langle \frac{\partial u_{n,k}(x)}{\partial k_y} \middle| \frac{\partial u_{n,k}(\bar{x})}{\partial k_x} \right\rangle - \left\langle \frac{\partial u_{n,k}(\bar{x})}{\partial k_x} \middle| \frac{\partial u_{n,k}(\bar{x})}{\partial k_y} \right\rangle \right) d^2\bar{k} =$$

density of states in k-space

$$= \frac{e^2}{h} E_x \oiint_{MBZ} \frac{1}{2\pi i} \left(\langle \ \rangle - \langle \ \rangle \right) d^2\bar{k}$$

But according to the experimental result the integral must be integer at the plateaux of the transverse $\sigma_{xy} = j_y / E_x$

$= G_{HALL} = \frac{I}{V_{HALL}} = n e^2/h$

This integer n_{ch} is called the Chern number

To show that the Chern number is integer we use the Stokes theorem:

$$\langle \; | \; \rangle - \langle \; | \; \rangle =$$

$$= \left[\nabla_k \times \langle u_{n\bar{k}}(\bar{x}) | \nabla_k | u_{n\bar{k}}(\bar{x}) \rangle \right|_z :=$$

$$= \left[\nabla_k \times \underbrace{A_{Berry,n}(\bar{k})} \right]_z$$

where $\nabla_k = \frac{\partial}{\partial \bar{k}}$, and z is the 3^d component.

The vector:

$$\bar{A}_{Berry,n}(\bar{k}) \equiv \langle U_{n,k}(\bar{x}) | \nabla_k | U_{n,k}(\bar{x}) \rangle$$

is called the Berry connection

By the Stokes theorem if the integrand is continious

$$\sigma_{xy} = \frac{\dot{j}_y}{E_x} = \boxed{\frac{e^2}{h} \cdot \frac{1}{2\pi i} \oint_{\text{contour around MBZ}} \bar{A}_{Berry,n}(k) \cdot d\bar{k}}$$

$$\boxed{:= \frac{e^2}{h} \cdot \frac{1}{2\pi i} \cdot \varphi_{Berry,n}} \leftarrow \text{Berry PHASE}$$

Lets go back where we started: our quest for topology in physics:

$$\boxed{Q_I(z) = \frac{1}{2\pi i}\int_0^t \frac{dz(t)/dt}{z(t)}\,dt}$$

Lets confirm the Q_I is the quantity we want:

$$Q_I(z) = \frac{1}{2\pi i}\int_0^1 d/dt\,\big(\ln(z(t))\big)\,dt =$$

$$= \frac{1}{2\pi i}\ln(z(t))\Big|_0^1 = \frac{1}{2\pi i}\ln\left|\frac{z(1)e^{i\varphi(1)}}{z(0)e^{i\varphi(0)}}\right|$$

$$= \frac{1}{2\pi i}\cdot\left(\ln\left(e^{i\varphi(1)}\right) - \ln e^{i\varphi(0)}\right) =$$

$$= [\varphi(1) - \varphi(0)]/2\pi$$

if $\varphi(t)$ is continuous, i.e. no jump from $2\pi \to 0$ after the turn $\Rightarrow$ $Q_I(t)$ gives the number of turns!

notice if the phase difference $\varphi(1) - \varphi(0) = 2\pi\cdot n$ ← integer

$\Rightarrow$ Q_I is integer!

Let's compare this to σ_{xy}:

$$\sigma_{xy} = \frac{j_y}{E_x} = \frac{e^2}{h}\cdot\frac{1}{2\pi i}\oint_{\text{magnetic } B\,z} A_{Berry,n}(\bar{k})\,d\bar{k}$$

$$= \frac{e^2}{h}\,\frac{1}{2\pi i}\cdot\varphi_{Berry,n} =$$ a number of turns around a singularity in k-space in units of e^2/h

e.g.