INTEGRATED HALL EFFECT
Quantum Hall Effect.

Let's place our material into a high mag. field.

Recall from the lecture on $T$ we derived that

$$j^m = \sigma j^m = \mathbf{E} \mathbf{v} \rightarrow \mathbf{E}_m = \mathbf{B} \mathbf{v}_m$$

or for the case of $\mathbf{B} = \mathbf{B} \hat{z}$ we have

$$\frac{1}{nqe} \propto j'y'$$

$B$ can be used to measure carrier density $n$ and the sign of carriers, $\phi_0$.

Now let's place a 2D electron gas and do the same measurement.

This looks completely different!!!

---

**Figure 5**: Plots of Hall and longitudinal resistivities as a function of magnetic field. Plateaus in Hall resistivity at several fillings are visible. After Ref. [28]
First we see a set of plateaux in which $R_H = \frac{V_y}{I_x} = \rho_{xy}$ is quantized in units of
\[ \frac{\hbar}{2e^2} = \frac{1}{2} \frac{\hbar}{e^2} \] (von Klitzing, Dorda, Pepper).

This feature is universal, independent of
material, impurities etc.

But according to the plateaux $\rho_{xx} \to 0$ (dissipationless): drops by 13 orders.

In 1982 Tsui, Störmer and Gossard showed that in the presence of disorder $\rho_{xx}$ becomes fractional = FQHE

which involves strong electronic correlations. Fermions condense into quantum states with excitations carrying fractional quantum numbers

\[ \frac{\hbar}{2e^2} = \text{quantum units of resistance}. \]

\[ R_H = \frac{B}{h e^2 c} = \frac{\hbar}{e^2} \frac{B}{h c} < \frac{1}{e^2} \]

\[ = \frac{\hbar}{e^2} \frac{1}{N_e} = \frac{\hbar}{e^2} \frac{N_0}{N_e} \]

\[ \hbar \frac{N}{A} \]

\[ \text{Topologically ordered phases.} \]
Here \( A \) is the area of the sample,

\[ N_e = \frac{n A}{\Phi} \quad \text{# of electrons} \]

\[ \Phi = BA \quad \text{flux through the system} \]

\[ \Phi_0 = \frac{hc}{e} \quad \text{flux quantum} \]

\[ \frac{N_e}{N_\Phi} = \text{# of the flux quanta} = \text{Landau - level filling factor} \]

**Why do we care about 2D?**

\[ R = \text{Resistivity for the d-dimensional system} \]

\[ = \rho \frac{L}{(2 - d)} \]

\[ \text{Resistance} \]

\[ \text{if } d = 2 \]

\[ R = \rho \quad \text{in 2D only} \]

Meaning that this quantity is a scale invariant!

In 2D \([\rho] = [\rho] = \left[ \frac{1}{L^2} \right] \) the same units.

But \( R \) is still geometry dependent

\[ R = \rho \frac{L}{w} \]

**But** \( R \) **is still geometry dependent**

\[ R_{xy} = \frac{V_y}{I_x} \]
Notice $\frac{L}{W}$ describes geometry and not its size.

For Hall = $R_{yx} = \frac{V_y}{I_x} = \frac{E_y}{I_x} = \frac{E_y}{(I_x/W)}$

$= \frac{E_y}{I_x} = \frac{I_y}{x} \propto \text{no} \frac{W}{L} \text{at all!}$

Moreover since it's dissipationless transport the location of contacts is unimportant.

As there is no voltage drop it's isopotential.

Notice, the lack of dissipation is b/c of a quantum effect $\Rightarrow$ quenching of kinetic energy by strong mag. field.

In short, Hall resistance = Hall resistivity.

Why disorder is important?

Strangely but true, the universality of TQHE is due to disorder?! 

This is the case of Anderson localization which in 2D any amount of disorder should cause the localization.
Specifically, in the absence of disorder, EDEG is **translationally invariant** (ignore the periodic potential of the xtal!) if we use the frame which moves with \(-v\) w.r.t. Lab frame:

\[ \mathbf{J} = -ne\mathbf{E} \]

In the lab frame:

\[ \mathbf{E} = 0 \quad \overline{\mathbf{B}} = \overline{\mathbf{B}}_z \]

In the moving frame:

\[ \overline{\mathbf{E}} = -\frac{1}{c} \mathbf{v} \times \overline{\mathbf{B}} \]

(Coordinate transformation)

\[ \overline{\mathbf{B}} = \overline{\mathbf{B}}_z \]

We need to cancel this Lorentz force by this electric field:

\[ \overline{\mathbf{E}} = \frac{\mathbf{B}}{ne\epsilon_0} \mathbf{J} \times \overline{\mathbf{B}} \]

Recall:

\[ E_{\mu} = f_{\mu
u} T^{\nu} \Rightarrow \]

\[ \mathbf{p} = \frac{\mathbf{B}}{e\epsilon_0} \begin{pmatrix} p_{0x} \\ 0 \\ -p_{0y} \end{pmatrix} \Rightarrow \mathbf{p} = \frac{ne\epsilon_0}{\overline{\mathbf{B}}} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

Since \( s_{xx} = 0 \) it's an insulator,

but \( p_{xx} = 0 \Rightarrow \) perfect conductor. (Normally if \( s_{xy} = 0 \), \( p_{xx} \to \infty \))
But $\delta_{xy} = -\frac{n e c}{\beta} \neq 0$, so it's not a superconductor.

Notice the only info about 2DEG we need is $\hbar$, if there is no disorder.

So if we had a perfect sample we wouldn't find any new physics?! (b/c of translational invariance)

Thus we need disorder to kill the tr. invar.

**Semi-Classical Transport:**

In 2D: \[ \begin{align*}
 m \ddot{x} &= -\frac{eB}{c} \dot{y} \\
 m \ddot{y} &= +\frac{eB}{c} \dot{x}
\end{align*} \]

\[ \implies \dot{x} = -\frac{c \dot{w}}{eB} x \]

\[ \implies \begin{cases} 
 x' = \frac{eB}{c} x \\
 \dot{w} = -\frac{eB}{c} w
\end{cases} \]

\[ \implies \begin{cases} 
 x = \frac{eB}{c} t \\
 \dot{w} = \frac{eB}{m c^2} \dot{w}
\end{cases} \]

or \[ \color{green}{w_{cyclotron} = \frac{eB}{mc}} \]

and $(x, y) = \mathbf{r} = R (\cos (\omega t + \phi), \sin (\omega t + \phi))$

\[ T = \frac{1}{w_c} \quad \text{independent of } R! \]

\[ \mathbf{r} = R w_c \]
This kind of motion is called \textit{isochronous} e.g., in oscillator where $T$ is indep. of amplitude.

Let's review the approach to canonical momentum: \textit{Classical Mech.}

$$\dot{x} = \frac{1}{2} m \dddot{x} \dot{x}^2 - \frac{\theta}{c} \dot{x} \dot{A} \dot{x} A$$

$\dot{A} = \text{vector potential at } x \mu$

The eqn. of motion

$$\frac{\partial L}{\partial \dot{x}^\mu} = -\frac{e}{c} \dot{x}^\mu \partial_\nu A^\nu$$

$$\frac{\partial L}{\partial \dot{t} \dot{x}^\mu} = m e \dot{x}^\mu - \frac{e}{c} \dot{A}$$

The \textit{Euler - Lagrange} eqn. of motion

$$m \dddot{x}^\mu = -\frac{e}{c} \left[ \partial_\nu A^\nu - \partial_\nu A^\nu \right] \dot{x}^\mu$$

or by using $B = \nabla \times A$

$$B^\sigma = \epsilon^{\alpha \beta \gamma \sigma} \partial_\beta A_\gamma$$

or

$$m \dddot{x}^\mu = -\frac{Be}{c} \dot{x}^\mu$$

Once we have the Lagrangian, the canonical momentum

$$p^\mu = \frac{\delta L}{\delta \dot{x}^\mu} = m \dot{x}^\mu - \frac{\theta}{c} \dot{A} \dot{x} A$$
And the Hamiltonian

$$H (p, x) = \dot{x}^M p_M - \alpha (\dot{x}, x) =$$

$$= \dot{x}^M (m \dot{x}^M - \frac{e}{c} A^M) -$$

$$- \frac{1}{2} m \dot{x}^M \dot{x}^M - \frac{e}{c} \dot{x}^M A^M =$$

$$= \frac{1}{2} m \left( p^M + \frac{e}{c} A^M \right) \left( p^M + \frac{e}{c} A^M \right) =$$

$$= \frac{m}{2} \sum \pi^M \pi^M$$

Mechanical momentum

Hamilton equations:

$$\dot{x}^M = \frac{\partial H}{\partial p^M} = \frac{\pi^M}{m}$$

$$\dot{p}^M = -\frac{\partial H}{\partial x^M} = -\frac{e}{mc} \left( p^M + \frac{e}{c} A^M \right)$$

so

$$\sum \pi^M = m \dot{x}^M$$

Let's quantize it semiclassically. To make a big fat electron we introduce a wave packet made of Bloch waves: $\Psi_{R(t)}, k(t) (r, t)$
The packet must be large $\Delta \ell$ de Broglie so we can define a central $k(t)$ and $E(\ell)$ (no Heisenberg here!)

from semiclassical theory (see Ch. 8 of the main text)

\[
\begin{align*}
\dot{R}^\mu &= \frac{\partial \langle \Psi_{R,k} | H | \Psi_{R,k} \rangle}{\partial (\hbar K^\mu)} \\
\hbar \dot{K}^\mu &= -\frac{\partial \langle \Psi_{R,k} | H | \Psi_{R,k} \rangle}{\partial R^\mu}
\end{align*}
\]

This works for weak magnetic field and fast electrons.

e.g. $\hbar w_c \ll E_F$

or in the oscillator we have $\hbar w_c \sqrt{B}$

and $L = \sqrt{\frac{\hbar c}{eB}} \times \frac{2.57A}{\sqrt{B} \text{(Tesla)}} = \frac{2.57A}{1T} \approx 25 \text{ nm}$

\[L \approx 25 \text{ nm} \]

The physical meaning inside the area $L^2$
we have exactly 1 quantum of flux 
\[ \Phi_0 = \frac{\hbar}{e} \rightarrow \text{so the density of magn. flux} \]

\[ B = \frac{\Phi_0}{2\pi L^2} \]

And now for the quantum version in the strong field.

1st let's select a gauge for the vector potential, e.g., Landau gauge

\[ \vec{A}(\vec{r}) = -x\vec{B} \hat{y} \rightarrow \nabla \times \vec{A} = \frac{\vec{B}}{\epsilon} \]

So the potential points in \( \vec{y} \) but varies with \( x \)!

Also, the system is translationally invariant in \( y \)

so translation leaves \( \nabla \times \vec{A} \) invariant in \( y \) but not \( \vec{A} \) itself.

Next, we ignore the band structure due to the periodic potential, and we can write the Hamiltonian in the Landau gauge as:

\[ \sqrt{H} = \frac{1}{2m} \left( \frac{\vec{P}^2}{2} + \left( \frac{\vec{P}_y - \frac{e\vec{B}}{c} x}{\epsilon} \right)^2 \right) \]

\( b/c \vec{A} \) points in \( x \).
B/c system is translational in y, we separate variables as:

$$\psi_k(x, y) = e^{i k y} f_k(x)$$

and for H $$\psi_k = \xi_k \psi_k$$ or

$$\mathcal{H} f_k(x) = \xi_k f_k(x)$$

where

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{1}{2m} \left( \frac{h_k}{c} - \frac{eB}{c} \right)^2$$

$$= \frac{p_x^2}{2m} + \frac{1}{2} m \omega_c^2 (x - x_k)^2$$

$$\uparrow \quad = \sqrt{\frac{h \omega_c}{eB}}$$

Let's label $$k \omega_c^2 = \xi_k$$ we end up

$$\left[ \frac{p_x^2}{2m} + \frac{1}{2} m \omega_c^2 (x - x_k)^2 \right] f_k(x) = \xi_k f_k(x)$$

this is 1D eqn for a harmonic oscillator with $$\xi_{k,n} = (n + \frac{1}{2}) \hbar \omega_c$$
The eigenstate is:

\[ \psi_{n,k}(r) = e^{i k y} H_n \left( \frac{x}{e - k e} \right) e^{-\frac{(x - k e)^2}{2e^2}} \]

this can be verified by the direct substitution.

These levels are called Landau levels.

*(no spin here included yet!)*
Ideal gas of electrons:

\[ E = \frac{\vec{p} \cdot \vec{p}}{2m_0} = \frac{p_x^2 + p_y^2}{2m_0} + \frac{p_z^2}{2m_0} = E_\parallel + E_\perp \]

Recall the density of states for 2D is \( \rho_{2D}(E) = \frac{m^*}{\hbar^2} (2D \text{ only}) \)

- Every energy level is degenerate
  - for each \( E \) we have many \( p_x \) and \( p_y \) such as \( p_x^2 + p_y^2 \)
  - for many \( n_x \) and \( n_y \)

in the plane \( \vec{B} \) electrons move on the circle of

\[ r_0 = \frac{m_0 v_0}{B} \]

\( B \neq 0 \) with \( \omega_c = \frac{eB}{m_0} \)

\( n \) the energy is quantized:

\[ E = E_\perp + E_\parallel = \frac{\hbar \omega_c}{2} \left( n + \frac{1}{2} \right) + \frac{p_z^2}{2m_0} \]

\( n = 0, 1, 2, \ldots \)

\( n_x, n_y = 0, 1, 2, \ldots \)
For energy $E_+$ we have only energies $\hbar \omega_c \left( n + \frac{1}{2} \right)$ separated by $\hbar \omega_c$.

For $N < n$ we get $E_n = \frac{p_x^2}{2m_0} = \frac{\hbar^2}{2m_0} \left( \frac{2\pi \hbar}{L_x} \right)^2 n^2$ 

huge # of states almost quasicontinuous.

See page 5

The # of $e^-$ in the band of size $\frac{\hbar \omega_c}{2}$

$N_L = \frac{v_{2D}(E)}{\hbar \omega_c} \cdot \frac{\hbar \omega_c}{\Delta E}$ = $\frac{S \sin \frac{\pi}{2} \cdot \hbar \omega_c}{\pi \hbar^2} = \frac{L_x \cdot L_y}{\pi \hbar^2} \cdot \text{motive}$

$N_L$ defines the degree of degeneracy of $E_+$ for $B \neq 0$.

For discrete values of $E_n^1$ in the quasiclassical approximation corresponds a specific trajectory; which depends on the quantum # $n$. Then our condition:

$\lambda_0 \ll \gamma_{\text{Bohr}}$ is equal $\hbar \omega_c \ll E_F$. 

\[ \lambda_0 \ll \gamma_{\text{Bohr}} \] is equal $\hbar \omega_c \ll E_F$.
To find the radius $R_B$, let's compare

$$E_{\text{classical}} = \frac{m_0 \omega_c^2 R_B^2}{2}$$

and

$$E = \hbar \omega_c (n + \frac{1}{2})$$

From this, we get

$$R_B = \sqrt{\frac{2 \hbar}{m_0 \omega_c} (n + \frac{1}{2})} = \sqrt{\frac{2 \hbar}{eB} (n + \frac{1}{2})}$$

So for the electron to go from the orbit $n$ to $n+1$, it needs to get a nice $\hbar \omega_c$.

For the same $n$, $L^2$ get the same $R_B$.

But $n_2$ and $p_2$ can be different.

**Let's include spin**

For electron with mag. moment $\mu_B = e \hbar / 2m_0 \omega_c$.

Its energy in $B = -\gamma \beta i$ with the spin we split a Landau level into two sublevels depending on $\mu \uparrow \downarrow$ $B$ or $\mu \downarrow \uparrow$ $B$.

$$E(n_s, k_z) = \hbar \omega_c (n + \frac{1}{2}) + \frac{1}{2} \hbar \omega_c k_z^2$$

$s = \pm 1$

$s = \pm 1 \Rightarrow "+" \text{ state if the lowest level }$ $\downarrow$

$-1 \ldots "-" \ldots$ $\uparrow$

Note: Spin removes the Landau degeneracy for the same $n$ we have $n_s = +1 \quad n_s = -1$.

(See page 8)
This note should be on the previous page:
The distance between LLS is very small
i.e. $t_{\omega c} = 0.18 \, (\text{meV/T}) \cdot \frac{B \cdot \text{we}}{m^*}$

so e.g. if $m^* = \text{we}$ and $B = 20 \, T$
we get $t_{\omega c} = 0.36 \, \text{meV}!$
Extremely small # we cannot measure experimentally, e.g. ARPES > 1-2 meV even at 10K.

Thus we must:

- reduce disorder to the absolute min
- have a system with very light $e^-$
  e.g. $m^* \text{GaAs} = 0.067 \, \text{meV}$
- in meters we cannot observe the QHE b/c life-time of $e^-$ < time for a full cyclotron orbit.

No orbit no quantization.
Each state on the parabola is strongly degenerate.

Spin degeneracy is only absent for $0^-$. Since $E$ continuously depends only on $p_z$, it looks like we have a quasi-1D system!

**Distribution of Electrons in $p$-Space**

Assume for have a single zone metal with a spherical Fermi surface. If $B=0$ all the states are inside the sphere and occupy $(2\pi)^3$. So we mark the $p_y$ points separated by $2\pi$. The maximum circle is $p_F = \sqrt{2m^*E_F}$. For any other $X$-section the states fill up the circle of $\sqrt{p_F^2 - p_z^2}$; as $p_z \to p_F$ the radius goes $\to 0$. 

\[ \begin{align*}
E_{\text{parabola}} &= \frac{3}{2} \hbar c + \frac{p_z^2}{2m_0} \\
E_{\text{states}} &= \frac{5}{2} \hbar c \\
& \quad \text{for } n = 2 \\
& \quad \text{for } n = 1 \\
& \quad \text{for } n = 0
\end{align*} \]
The uniform distribution of states with $p_x, p_y, p_z$ corresponds to $E = E(p_x p_y p_z)$ when $0 < |p| < p_F$

Now we turn on $B$: in the plane $\hbar \omega_c (n + \frac{1}{2})$

for $p_z = \text{const}$, to find the radius $p_n$

we write down $E_{\text{class}} = \frac{p_x^2 + p_y^2}{2m^*} = \frac{p^2}{2m^*}$

$E_{\text{quantum}} = \frac{1}{2} \hbar \omega_c (n + \frac{1}{2})$

$\Rightarrow p_n = \sqrt{2m^* \hbar \omega_c (n + \frac{1}{2})}$ (see fig. 6 in page 8)

In other words: all states which we had confined inside the orbits with a radius $p_n$, $n=0, 1, 2, \ldots$, now collapse on the circles see fig. a vs. b in page 8

Note the area in a) $\pi (p_{n+1}^2 - p_n^2) =$

Except for $0^-$ state:

$\pi p_0^2 = \pi m^* \hbar \omega_c$

So for each allowed orbit we have the same $#$ of $e^{-}$ $N_e = \frac{m^* \hbar \omega_c}{\pi \hbar^2}$

Degeneracy of those $p_n$ orbits is the same as the discrete Landau levels

Note since $p_n$ is independent of $p_z$ all
To be continued.

Topological p.o.v. for IQHE

- Orbits are of the same radius cylinder
- With increasing Pe, the length of cylinders depends on its length within cylinder filled up byPe on each cylinder
Topological properties of IQHE

Global geometrical properties of an object in the mathematical space.

E.g. K-space for the electron in the Hilbert space.

The goal is to classify objects based on geometrical properties:
- bending, stretching are not
  poking holes and glue ins is not!

E.g., how many times the loop widens up before it enclose the point P.

"Answer: 2 times."

Let's try this mathematically:
1st we define the function
\[ z(t), \, t \in [0, 1], \, t \in \mathbb{R} \]
\[ z \in \mathbb{C}, \, \text{as usual} \]
\[ z(t) = |z(t)| \cdot e^{i \varphi(t)} \]

Now we can define the integral:

\[ Q_I(z) = \frac{1}{2\pi i} \int_0^t \frac{d\bar{z}(t)}{\bar{z}(t)} \]  

Let's confirm that \( Q_I \) is the quantity we want:

\[ Q_I(z) = \frac{1}{2\pi i} \int_0^t \frac{d}{dt} \left( \ln(\bar{z}(t)) \right) dt = \]

\[ = \frac{1}{2\pi i} \ln(\bar{z}(t)) \bigg|_0^t = \frac{1}{2\pi i} \ln \left( \frac{\bar{z}(t) e^{i \varphi(t)}}{\bar{z}(0) e^{i \varphi(0)}} \right) \]

\[ = \frac{1}{2\pi i} \cdot \left( \ln \left( e^{-i \varphi(t)} - \ln e^{i \varphi(0)} \right) \right) = \]

\[ = [\varphi(t) - \varphi(0)]/2\pi \]

if \( \varphi(t) \) is continuous, i.e. no jump from \( 2\pi \rightarrow 0 \) after the turn \( \Rightarrow \) \( Q_I(z) \) gives the number of turns \( \Sigma \).
Most important we classify all possible paths in $\mathbb{R}^d$.

Obviously $Q_i(x)\in\mathbb{Z} = \text{is called } Q_i(x) \text{ is a } \mathbb{Z} - \text{type topological invariant.}$

Let’s apply this concept to IQHE

For this purpose we rederive Hall conductivity tensor quantum-mechanically in Kubo approximation.

1) Bloch state in a solid is $\psi_{nk}(x) = U_{nk}(x)e^{ikx}$
   - $n$ - band index
   - $k$ - wave vector

2) Apply 1st order perturbation theory in the weak electric field $\vec{E} = E_x \cdot \hat{x}$
   - the electric potential ($B=0$)
     $$\phi_{el}(x) = E_x \cdot x = -i \frac{d}{dk_x} \cdot eE_x$$

The perturbed w.f.

$$|n\rangle = |h\rangle - \sum m \neq h \frac{|m\rangle \langle m| eE_x \frac{d}{dk_x} |h\rangle}{E_{ho} - E_{mo}}$$

Let’s determine the velocity in y direction
\[ \psi_y = \frac{-i}{\hbar} \sum_{m \neq n} \langle \text{no} | V_y | \text{no} \rangle \frac{d}{dx} \langle \text{no} | d \phi | \text{no} \rangle + \text{h.c.} \]

\[ \langle \text{no} | V_y | \text{no} \rangle = \frac{-i}{\hbar} \left( \langle \text{no} | H_y | \text{no} \rangle - \langle \text{no} | V_y | \text{no} \rangle \right) \]

\[ \langle \text{no} | V_y | \text{no} \rangle = \frac{-i}{\hbar} \left( \langle \text{no} | H_y | \text{no} \rangle - \langle \text{no} | V_y | \text{no} \rangle \right) \]

\[ \langle \text{no} | H_y | \text{no} \rangle = \frac{-i}{\hbar} \langle \text{no} | H | \text{no} \rangle \cdot (E_{\text{no}} - E_{\text{no}}) \]

\[ \langle \text{no} | V_y | \text{no} \rangle = \frac{-i}{\hbar} \left( \langle \text{no} | H | \text{no} \rangle \cdot (E_{\text{no}} - E_{\text{no}}) \right) \]

for all \( m \neq n \). Insert \( \frac{\partial}{\partial k_x} \) into \( \psi_y \)

\[ \psi_y = \left\{ \frac{-i}{\hbar} \langle \text{no} | V_y | \text{no} \rangle + \frac{i e X}{\hbar} \right\} \sum_{m \neq n} \langle \text{no} | \frac{\partial}{\partial k_x} | \text{no} \rangle \cdot (E_{\text{no}} - E_{\text{no}}) \]

\[ \langle \text{no} | \frac{\partial}{\partial k_x} | \text{no} \rangle \]

from \( \frac{\partial}{\partial k_x} \langle \text{no} | V_y | \text{no} \rangle \langle \text{no} | H_y | \text{no} \rangle (E_{\text{no}} - E_{\text{no}}) = 0 \)

\[ \sum_{m \neq n} \sum_{m_0 \neq n_0} 1 \]

\[ \langle h_0 | \frac{\partial}{\partial k_x} | h_0 \rangle = \frac{-i}{\hbar} \langle h_0 | \frac{\partial}{\partial k_x} | h_0 \rangle (E_{h_0} - E_{h_0}) = 0 \]
So finally, we have:

\[
\psi_y = \frac{i e E_x}{\hbar} \left( \langle \frac{\partial \phi_0}{\partial k_y} | \frac{\partial \phi_0}{\partial k_x} \rangle - \langle \frac{\partial \phi_0}{\partial k_x} | \frac{\partial \phi_0}{\partial k_y} \rangle \right) - \langle \frac{\partial u_{nk}(\vec{x})}{\partial k_x} | \frac{\partial u_{nk}(\vec{x})}{\partial k_y} \rangle \right) \neq 0 \text{ maybe}
\]

and since the plane wave part in \[1\psi_0 = u_{nk}(\vec{x}) e^{i k x} \]
doesn't contribute, we finally get:

\[
\psi_y = \frac{i e E_x}{\hbar} \left( \langle \frac{\partial u_{nk}(\vec{x})}{\partial k_y} | \frac{\partial u_{nk}(\vec{x})}{\partial k_x} \rangle - \langle \frac{\partial u_{nk}(\vec{x})}{\partial k_x} | \frac{\partial u_{nk}(\vec{x})}{\partial k_y} \rangle \right)
\]

Linear response theory based on Kubo formalism.

To get current \[\mathbf{j}_y\] in the electric field \[E_x\], we add up all the contributions from all occupied states \[u_{nk}(\vec{x})\].

\[\mathbf{j}_y = -e \psi_y\]
The transverse current $\neq 0$ if $\frac{\partial U}{\partial k_x}$ and $\frac{\partial U}{\partial k_y}$ are different! and contribution from different $k$, should not cancel.

Now we need to prove that the same Bloch w.f. works for a magnetic field.

Recall the translation operator ($\theta = 0$)

$$ T(R_n) = e^{i R_n \cdot \mathbf{D}} $$

$$ T(R_n) \cdot f(\mathbf{x}) = f(\mathbf{x} + R_n) = T(R_n) \text{ commutes with } V(\mathbf{x}) \Rightarrow T(V(\mathbf{x})) = V(\mathbf{x} + R_n) $$

it also commutes with $\mathbf{D}^n$, $n = 1, 2, \ldots$ $= V(\mathbf{x})$

$\Rightarrow$ it commutes with $H = -\frac{\hbar^2}{2m} \mathbf{D}^2 + eV(\mathbf{x})$

$\Rightarrow$ eigenstates of $H$ and $T$ are common $\Rightarrow$ exactly Bloch functions

Now we apply an external mag. field
\[ \hat{H}_B = \frac{1}{2m} \left( \imath \hbar \partial + e \hat{A}(x) \right)^2 + e \hat{V}(x) \]

where \[ \hat{A}(x) = -\frac{1}{2} \left( \frac{\hat{x} \times \hat{B}}{\left| \hat{x} \times \hat{B} \right|} \right) \]

\[ \uparrow \text{ in symmetric gauge} \]

Since \[ A(x) \neq A(x + R_n) \]

\[ T(R_n) \text{ doesn't commute with } \hat{H}_B, \]

but \[ \hat{A}(x) \]

\[ \text{new} \]

\[ T_B(R_n) = e^{-\frac{e \hat{A}(x)}{\imath \hbar}} \]

\[ \text{no field} \]

will commute with \[ \imath \hbar \partial + e A(x) \]

\[ \text{Show this!} \]

But now the problem is:

\[ T_B(R_n) V(x) = e^{\frac{e \hat{A}(x)}{\imath \hbar}} \left( \frac{\hat{x} \times \hat{B}}{\left| \hat{x} \times \hat{B} \right|} \right) \]

\[ T_B(R_n) V(x + R_n) \]

\[ V(x + R_n) \]

Imagine we now move in the loop by applying the operator \( T_B(R_n) \) many times.

\[ \text{area } \tilde{A} \]

\[ \text{size } A \]

\[ i \frac{e B}{\hbar} \cdot \tilde{A} = i \frac{2 \pi e B}{\hbar} \tilde{A} = \text{integer} \]

\[ \int \hat{A}(x) \hat{x} = \int \int \nabla x \cdot \hat{A}(x) \ d\hat{A} = \int \int \beta \ d\hat{A} = 1 \beta \cdot \hat{A} \cdot \text{sgn} (\beta \hat{A}) \]
b/c \( e^{i 2\pi} = 1 \) the phase will vanish if \( \tilde{A} \) contains an even number (!) of magnetic flux quanta

\[
2\pi i \quad \frac{B A}{\phi} = n \quad \text{or} \quad \phi = 2\pi n = \text{integer even}
\]

Note the flux quantum is independent of gauge \( \tilde{A}(\tilde{x}) \)

\( \Rightarrow \) we can now define a new unit cell that contains an even number of flux quanta. The new unit cell is called the \textit{magnetic u.c.} with lattice vectors \( \mathbf{R}_n, \mathbf{b} \). The Schrödinger equation commutes with \( \hat{T}_B(\mathbf{R}_n, \mathbf{b}) \) and hence Bloch w.f. still good for \( B \neq 0 \).

\( \Rightarrow \) But b/c of the extra phase

\[
e^{-\frac{2\pi i}{\hbar} \mathbf{A}^{\wedge}(\tilde{x})} \quad \text{in} \quad \hat{T}_B(\mathbf{R}_n, \mathbf{b}) \quad \hat{\Theta}_{n,k}(\tilde{x})
\]

\( \psi_n(\tilde{x}) \) is not simple but \( \psi_{n,k} = \psi_n(\tilde{x}) \cdot e^{i \Theta_{n,k}(\tilde{x})} \)

with magnetic unit cell \( \frac{d \Theta_{n,k}(\tilde{x})}{d\tilde{s}} = -2\pi B \philorenge
Finally we calculate

\[ j_y = -e \oint \frac{1}{(2\pi)^2} \hat{g} \left( \mathbf{k} \right) \, d^2 \mathbf{k} = \]

\[ \text{Mag. B}^2 \]

\[ = -e \oint \frac{1}{(2\pi)^2} \frac{i \hbar E_x}{\hbar^2} \left( \left\langle \frac{\partial U_{n,k}(\mathbf{x})}{\partial k_x} \right\rangle \frac{\partial U_{n,k}(\mathbf{x})}{\partial k_x} \right) \, d^2 \mathbf{k} = \]

\[ = -e^2 \frac{E_x}{h} \oint \frac{1}{2\pi i} \left( \left\langle \mathbf{k} \right\rangle - \left\langle k_y \right\rangle \right) \, d^2 \mathbf{k} \]

But according to the experimental result, the integral must be integer at the plateaux of the transverse \( G_{xy} = j_y/E_x \)

\[ = e - \text{HALC} = \frac{e}{\text{HALC}} = \hbar e/\hbar \]
This integer \( n_{\text{ch}} \) is called the Chern number.

To show that the Chern number is integer we use the Stokes theorem:

\[
\langle 1 | > < 1 | > = \left[ \nabla_k \times \langle U_n \tilde{e} (\tilde{x}) | D_k | U_n \tilde{e} (\tilde{x}) \rangle \right] \bigg|_{z} \bigg|_{\tilde{z}} = [\nabla_k \times A_{\text{Berry}, n} (\tilde{z})] \bigg|_{z}
\]

where \( \nabla_k = \frac{\partial}{\partial k} \), and \( z \) is the 3\( \text{d} \) component.

The vector:

\[
\tilde{A}_{\text{Berry}, n} (\tilde{z}) = \langle U_n \tilde{k} (\tilde{x}) | D_k | U_n \tilde{k}(\tilde{x}) \rangle
\]

is called the Berry connection.

By the Stokes theorem if the integrand is continuous

\[
\delta_{xy} = \frac{\text{J}_y}{E_x} = \frac{e^2}{\hbar} \cdot \frac{1}{2\pi i} \oint \tilde{A}_{\text{Berry}, n} (\tilde{x}) \cdot d\tilde{x}
\]

\[
\phi_{\text{Berry}, n} = \frac{e^2}{\hbar} \cdot \frac{1}{2\pi i} \oint \tilde{A}_{\text{Berry}, n} (\tilde{x}) \cdot d\tilde{x} \text{ contour around MBZ}
\]

\[
\phi_{\text{Berry}} \rightleftharpoons \text{Berry PHASE X}
\]
Let's go back where we started: our quest for topology in physics.

Now let's consider:

\[
Q_I(z) = \frac{1}{2\pi i} \int \frac{dz(t)/dt}{z(t)} \, dt
\]

Let's confirm this is the quantity we want:

\[
Q_I(z) = \frac{1}{2\pi i} \int \frac{dz(t)}{z(t)} = \frac{1}{2\pi i} \ln \left( \frac{z(t)}{z(0)} \right) = \frac{1}{2\pi i} \left( \ln(e^{i\phi(t)}) - \ln(e^{i\phi(0)}) \right) = \frac{\phi(t) - \phi(0)}{2\pi}
\]

If \( \phi(t) \) is continuous, i.e. no jump from \( 2\pi \) to 0 after the turn, \( Q_I \) gives the number of turns!

Let's notice if the phase difference

\[
\phi(t) - \phi(0) = 2\pi n
\]

\( n \) integer

\( \Rightarrow Q_I \) is integer!

Let's compare this to \( \sigma_{xy} \):

\[
\sigma_{xy} = \frac{jy}{Ex} = \frac{e^2}{h} \cdot \frac{1}{2\pi i} \int_{\text{magnetic}} A_{\text{Berry}, n}(\vec{k}) \, d\vec{k}
\]

\( = \frac{e^2}{h} \cdot \frac{1}{2\pi i} \cdot \phi_{\text{Berry}, n} = \text{a number of turns around a singularity in } k\text{-space in units of } \frac{e^2}{h} \)