Here is a very interesting observation.

3D crystal: 2D electron liquid/semiconductor interfaces e.g. GaAs

Based on the cylindrical symmetry

\[ p_{xx} = p_{yy}, \quad p_{xy} = p_{yx} \]

From classical mechanics (Debye theory) \( p_{xy} \sim B \)

\[ 2E_y = 2eB \quad j = e\nu n \quad \Rightarrow \quad p_{xy} = \frac{E_y}{J_x} = \frac{B}{E_y} \]

Very useful but nothing too spectacular.

Quantum Hall effect and its connection to quantum electrodynamics.

At low T and very very clean samples the 2DEG don't follow \( p_{xy} \sim B \)!

Instead, it shows a series of very strange plateaus. On the plate \( \delta_{xy} = \frac{1}{n} \), quantities \( \frac{e^2}{\hbar} n = 1, 2, 3, \ldots \) and \( p_{xx} = 0 \)

These plateaus are known as the \( Q = \frac{h}{e} \).
Interestingly, the value of conductivity can be expressed in terms of the fine structure constant:

\[ \alpha = \frac{e^2}{\hbar c} \]

\[ 6 \approx \frac{\hbar e^2}{4 \pi \hbar} \Rightarrow \hbar \alpha \ll \]

The fine structure constant \( \alpha \) measures the strength of quantum electrodynamics:

\[ \frac{\alpha}{\kappa} \]

\[ \text{potential energy} \]

\[ \text{potential energy} \]

\[ \text{kinetic energy} \]

If in our universe \( \alpha \approx 0 \) then electrons will not interact and in fact there will be no photons (no light - just imagine this).

If \( \alpha \) is large the universe will be made of very strongly entangled matter which will make the presence of life impossible as we know it.

However, \( \alpha \approx \frac{1}{137} \) which is just 1% of the kinetic energy and we live in the world with interactions \( \equiv \text{photons} \equiv \text{light} \) and yet we can do the perturbation theory.

Consider just kinetic energy as the unperturbed term, then in the 1st approx

\[ O(\alpha) \sim 1\% \quad O^2(\alpha) \sim 10^{-8} \]

Amazing but a condensed matter experiment can be as accurate as high energy physics in defining \( \alpha \).
QED: 
\[ \chi = \frac{e^2}{\hbar c} \sim \frac{1}{137} \]

CP: \[ \chi_{\text{CP}} = \frac{e^2}{\hbar U_{\text{Fermi}}} \] for a typical solid

\[ U_F \sim \frac{1}{100} - \frac{1}{1000} \]

So \[ \chi_{\text{CP}} = \frac{e^2}{\hbar U_F} \sim \frac{1}{10} \]

So the perturbation theory doesn't work.

QHE and Topology.

Before we answer why \( \sigma_{xy} \) is quantized, let's try to think why \( \sigma_{xx} = 0 \).

Q: if \( \sigma_{xx} = 0 \) is it a superconductor? or "perfect metal"?

A: No, no, no. The material is a conductor! Insulator!

The material has 0 conductivity:

\[ j = \sigma E, \quad \sigma = \rho \frac{1}{\rho} \]

This is only true if \( j \) and \( E \) are in the same direction. However, more generally:

\[
\begin{pmatrix}
J_x \\
J_y \\
J_z
\end{pmatrix} =
\begin{pmatrix}
\sigma_{xx} & \ldots & \sigma_{x2} \\
\vdots & \ddots & \vdots \\
\sigma_{z2} & \ldots & \sigma_{zz}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix}
\]

if all but \( \sigma_{xx}, \sigma_{yy}, \sigma_{zz} \) are not zero

\[ j = \sigma E \] and \( \rho = \frac{1}{\sigma} \) but for 2D
\[
\begin{align*}
\begin{pmatrix}
    i_x \\
    j_y
\end{pmatrix} &= \begin{pmatrix}
    6_{xx} & 6_{xy} \\
    6_{yx} & 6_{yy}
\end{pmatrix} \begin{pmatrix}
    E_x \\
    E_y
\end{pmatrix} \quad \text{and the resistivity}\end{align*}
\]
\[
\begin{align*}
\begin{pmatrix}
    E_x \\
    E_y
\end{pmatrix} &= \begin{pmatrix}
    p_{xx} & p_{xy} \\
    p_{yx} & p_{yy}
\end{pmatrix} \begin{pmatrix}
    j_x \\
    j_y
\end{pmatrix} =
\end{align*}
\]
\[
\begin{align*}
\begin{pmatrix}
    6_{xx} & 6_{xy} \\
    6_{yx} & 6_{yy}
\end{pmatrix} &= \begin{pmatrix}
    p_{xx} & p_{xy} \\
    p_{yx} & p_{yy}
\end{pmatrix}^{-1} =
\end{align*}
\]
\[
= \frac{1}{p_{xx}p_{yy} - p_{xy}p_{yx}} \begin{pmatrix}
    p_{yy} & -p_{xy} \\
    -p_{yx} & p_{xx}
\end{pmatrix}
\]

Now, let's go to the platen where \( p_{xx} = p_{yy} = 0 \):
\[
\begin{pmatrix}
    p_{xx} & p_{xy} \\
    p_{yx} & p_{yy}
\end{pmatrix} = \begin{pmatrix}
    0 & p_{xy} \\
    p_{yx} & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
    6_{xx} & 6_{xy} \\
    6_{yx} & 6_{yy}
\end{pmatrix} = \frac{1}{p_{xx}p_{yy} - p_{xy}p_{yx}} \begin{pmatrix}
    0 & -p_{xy} \\
    -p_{yx} & 0
\end{pmatrix}
\]
\[
= \begin{pmatrix}
    0 & \frac{1}{p_{yx}} \\
    \frac{1}{p_{xy}} & 0
\end{pmatrix} \Rightarrow 6_{xx} = 6_{xy} = 0. \ \ \text{NO CONDUCTIVITY}
\]

But between the platens, the conductivity is non-zero and as such it is METAL.

\( p_{xx} \neq 0 \) and \( 6_{xx} \) and \( 6_{yy} \) \( \neq 0 \).

\text{SUMMARY OF EXP. FACTS ABOUT QHE.}
1. When we vary the external field,
   - we can turn system ins $\leftrightarrow$ metal $\leftrightarrow$ ins. $\leftrightarrow$ metal.
   - each insulating state corresponds to a plateau of $\rho_{xy}$ and the step between 2 neighboring plateaus is metallic.
   - transport for the metallic states is NOT universal, and changes from sample to sample.
   - The insulating state is universal.
     $\rho_{xx} = 0$ and $\sigma_{xx} = 0$ and $\sigma_{xy}$ is quantized.

**Why IQH state is insulator?**

**Landau levels.**

Let's solve this problem quantum mechanically. In 2D for a charge neutral particle $q = 0$, the Schrödinger equation:

$$i \frac{\partial \psi(x,y)}{\partial t} = \left[ \frac{1}{2m} \left( -i \hbar \frac{\partial}{\partial x} \right)^2 + \frac{1}{2m} \left( -i \hbar \frac{\partial}{\partial y} \right)^2 \right] \psi(x,y)$$

with $H = \frac{1}{2m} \left( -i \hbar \frac{\partial}{\partial x} \right)^2 + \frac{1}{2m} \left( -i \hbar \frac{\partial}{\partial y} \right)^2$.

Now let's include charge and its connection to $E$ and $B$. He we will use a minimal coupling which tells that we change momentum $\vec{p} \rightarrow \vec{p} + e\vec{A} / c$ $A =$ vector potential

and $i \frac{\partial}{\partial t} \rightarrow i \frac{\partial}{\partial t} - \frac{e}{c} \vec{A} \cdot \vec{E}$ $\phi =$ electric potential
\[
\left( \frac{i}{\hbar} \frac{\partial}{\partial t} - \frac{e}{c} \phi \right) \psi(x, y) = \left\{ \frac{1}{2m} \left( -i \hbar \frac{\partial}{\partial x} - \frac{e}{c} A_x \right)^2 + \frac{1}{2m} \left( -i \hbar \frac{\partial}{\partial y} - \frac{e}{c} A_y \right)^2 \right\} \psi(x, y)
\]

\[
H = \frac{1}{2m} \left( -i \hbar \frac{\partial}{\partial x} - \frac{e}{c} A_x \right)^2 + \frac{1}{2m} \left( -i \hbar \frac{\partial}{\partial y} - \frac{e}{c} A_y \right)^2 + \frac{e}{c} \phi
\]

Since e=0, we can set \( \phi = 0 \Rightarrow \nabla \times A = \frac{\partial}{\partial y} A_x - \frac{\partial}{\partial x} A_y = B
\]

From Maxwell's equations, we know that \( A \) is not observable i.e. for the fixed \( B \) field, \( A \) is not uniquely defined. \( \nabla \times A = B \Rightarrow A = A^1 + \nabla \phi \)

is also the vector potential for the same field. \( \nabla \times A = \nabla \times A^1 \). We can choose any \( A \) as long as \( \nabla \times A = \nabla \phi \). Now let's apply the magnetic field along \( z : B_z \). Then we can use 2 options:

\[
\begin{align*}
\frac{\partial}{\partial x} A_y &= B_z \\
\frac{\partial}{\partial y} A_x &= 0
\end{align*}
\]

\( \nabla \times A = B \). \( A_x = 0 \) and \( A_y = B_x \) (the Landau gauge) \( \frac{\partial}{\partial x} A_x = B_y \) and \( A_y = \frac{B_x}{2} \) (the symmetric gauge)

Energy spectrum
\[ H = \left[ \frac{\hbar^2}{2m} \left( -i \frac{\partial}{\partial x} \right)^2 + \frac{1}{2m} \left( -i \frac{\partial}{\partial y} - \frac{e}{\hbar} B y \right)^2 \right] \]

Schrödinger equation: \[ H \psi = E \psi \]

in the Landau gauge \([P_y, H] = 0\) therefore we can find common eigenstates for \(P_y\) and \(H\).

\[ \psi(x, y) = f(x) e^{-i k y} \]

\[-\frac{\hbar^2}{2m} f''(x) + \frac{1}{2m} \left( \frac{\hbar}{m} k_y - \frac{e}{\hbar} B x \right)^2 f(x) = \epsilon f(x) \]

(for \(P_y\) the eigenvalue \(\hbar k_y\)), let's rewrite it:

\[-\frac{\hbar^2}{2m} f''(x) + \frac{e^2}{2mc^2} B^2 \left( x - \frac{c t}{e B} k_y \right)^2 f(x) = \epsilon f(x) \]

\[-\frac{\hbar^2}{2m} f''(x) + \frac{k}{2} \left( x - x_0 \right)^2 f(x) = \epsilon f(x) \]

Where \(x_0 = \frac{c t}{e B} k_y\) \(k = \frac{e B^2}{mc^2}\) \(L = \frac{c t}{e B}\) \(x_0 = \frac{L^2 k_y}{mc^2}\) \(\epsilon = \frac{c t}{e B}\)

how this equation looks like the harmonic oscillator.

Recall from your Quantum class, the solution is:

\[
\psi_n(k_y, x, y) = \phi_n(x - x_0) e^{-i k y} \quad \text{with} \quad \epsilon_n = (n + \frac{1}{2}) \hbar \omega = (n + \frac{1}{2}) \hbar \sqrt{\frac{k}{\mu}} = (n + \frac{1}{2}) \hbar \sqrt{\frac{e B}{\mu c}} \]

so the electrons are quantized in \(x, y\) and have continuous translation along \(z\); so the total \(E\)

\[ E = \epsilon_n + \frac{\hbar^2}{2m} k_y^2 = (n + \frac{1}{2}) \hbar \omega + \frac{\hbar^2}{2m} k_y^2 \]
if we include the spin of the electron then each level will split into 2 sub-levels \( \pm \frac{1}{2} \) most since the energy spectrum is dramatically affected we need to see what happens to the electronic density of states.

We start with

\[ \frac{2\pi}{L_x} \rightarrow \frac{?}{?} = k_x \text{ and } k_y \text{ are quantized} \]

in units \( \frac{2\pi}{L_x} \text{ and } \frac{2\pi}{L_y} \)

Also recall \( x_0 = \frac{c}{eB} \), \( k_y = \frac{2\pi e^2}{L_y} \),

And the degeneracy of the level in 2D is:

\[ D = \frac{L_x}{x_0} = \frac{L_x L_y}{2\pi e^2} \text{ The total magnetic flux through the } x-y \text{ plane is } \Phi = \frac{HL_x L_y}{c} \text{ and the flux quantum: } \Phi_0 = \frac{\hbar e}{c} \Rightarrow \]

\[ D = \frac{\Phi}{\Phi_0} \Rightarrow \text{ the number of states } = \text{ number of the flux quanta in units of } \frac{\hbar e}{c} ! \]

The Physical Meaning:

After applying the magnetic field points distributed in the \( k_x-k_y \) area get spread out onto circles with energies concentric \( \frac{\hbar c}{2}, \frac{3\hbar c}{2}, \frac{5\hbar c}{2} \ldots \)

But the total number of states remain the same. To show this lets calculate the number of states per unit area per unit energy and no spin.

\[ g_c(E) = \frac{1}{L_x L_y} \frac{D}{\hbar c} = \frac{M}{2\pi \hbar^2} \]

which is the same as \( g(E) = 2D/E \) without magnetic field attached.

Now lets extend this to 3D:

Note \( k_z \) is still a good quantum number so we can plot quantum number \( E(k_z) \) vs. \( k_z \) as bands also known as Landau subbands
Overall for 3D we have:
\[ g_{3D}(E) = \frac{1}{(2\pi)^2} \left( \frac{2m_1}{\hbar^2} \right)^{3/2} \hbar^2 c \sum_n \left[ E - (n + \frac{1}{2}) \hbar \omega_c \right]^{-1/2} \]

For 2D this is very interesting

On the Landau Levels: electrons collapse into a set of cylinders where they increase their degeneracy and move on the circles with \( \omega_c \) frequency.

Moving electrons introduce flux through the cylinder with the flux \( \Phi \) equal to the \# of electrons in units of \( \Phi_0 = \frac{hc}{e} \).
Topology and insulators

Topology came from 3D real space but now moved to the Hilbert space. Def: if a manifold \( M_1 \) can be adiabatically transformed into \( M_2 \), their topology is the same.

To distinguish between them, we introduce an object called index (a topo index). The same topo object = the same index.

For 2D: \( \chi_M = \frac{1}{2\pi} \oint K ds \) (the Euler characteristic).

<table>
<thead>
<tr>
<th>Name</th>
<th>Image</th>
<th>Vertices</th>
<th>Edges</th>
<th>Faces</th>
<th>Euler characteristic: ( V - E + F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>![Tetrahedron]</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Hexahedron or cube</td>
<td>![Hexahedron or cube]</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Octahedron</td>
<td>![Octahedron]</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>![Dodecahedron]</td>
<td>20</td>
<td>30</td>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>![Icosahedron]</td>
<td>12</td>
<td>30</td>
<td>20</td>
<td>2</td>
</tr>
</tbody>
</table>

See Wikipedia on the Euler characteristic
<table>
<thead>
<tr>
<th>Name</th>
<th>Image</th>
<th>Euler characteristic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interval</td>
<td><img src="image" alt="Interval" /></td>
<td>1</td>
</tr>
<tr>
<td>Circle</td>
<td><img src="image" alt="Circle" /></td>
<td>0</td>
</tr>
<tr>
<td>Disk</td>
<td><img src="image" alt="Disk" /></td>
<td>1</td>
</tr>
<tr>
<td>Sphere</td>
<td><img src="image" alt="Sphere" /></td>
<td>2</td>
</tr>
<tr>
<td>Torus (Product of two circles)</td>
<td><img src="image" alt="Torus" /></td>
<td>0</td>
</tr>
<tr>
<td>Double torus</td>
<td><img src="image" alt="Double torus" /></td>
<td>-2</td>
</tr>
<tr>
<td>Triple torus</td>
<td><img src="image" alt="Triple torus" /></td>
<td>-4</td>
</tr>
<tr>
<td>Real projective plane</td>
<td><img src="image" alt="Real projective plane" /></td>
<td>1</td>
</tr>
<tr>
<td>Möbius strip</td>
<td><img src="image" alt="Möbius strip" /></td>
<td>0</td>
</tr>
<tr>
<td>Klein bottle</td>
<td><img src="image" alt="Klein bottle" /></td>
<td>0</td>
</tr>
<tr>
<td>Two spheres (not connected)</td>
<td><img src="image" alt="Two spheres" /></td>
<td>(2 + 2 = 4)</td>
</tr>
<tr>
<td>(Disjoint union of two spheres)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Three spheres (not connected)</td>
<td><img src="image" alt="Three spheres" /></td>
<td>(2 + 2 + 2 = 6)</td>
</tr>
<tr>
<td>(Disjoint union of three spheres)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
we locally fit the surface for to a particular
curve and the $1/k$ is the local curvature $k$
Among all the curvatures the largest and
the smallest curvatures $k_1 = \frac{1}{k_1}$ and $k_2 = \frac{1}{k_2}$
are the principal curvatures. The gaussian
curvature
For sphere $k = k_1 = k_2$.
For a saddle point $k_1 > 0$ and $k_2 < 0$.

$\Rightarrow k = k_1 \cdot k_2 < 0$

The for any well-behaved 2D surface
the $\int K ds = 2\pi \cdot n = X_M$ so it's quantized!
Also for the surfaces with the same topology $X_M$
are the same.

The for any orientable closed surface $X_M$

is always an even integer.

Orientable means we can distinguish two
sides of the surface. If we cannot
then $X_M$ is ODD (e.g. a Möbius strip).

$X_M$ is a topological index only when the
surface has no boundaries. Otherwise, it's
not quantized and not topological.
Other topological properties:

1. Topology and handles
   \( X_m \) is related to the genus of a surface (or many-fold)
   \[ X_m = 2(1 - g) \]
   where \( g \) is the number of handles of the object.

2. Sphere \( g = 0 \) torus \( g = 1 \)
   a coffee mug = a donut, or
   3-handles cup = a pretzel = a triple torus

3. \( X_m \) and polyhedrons
   To define \( X_m \) we can draw a grid of polyhedral edges
   \[ X_m = V - E + F \]
   corners faces e.g. since a sphere has \( X_m = 2 \)
   we know that

4. Topology and hair vertex
   \( V - E + F = 2 \)
   If we draw a vector at each point of the surface
   we get a vector field, which may have vortices.
   Now we define a

\textbf{ignore!}

\textbf{uaiuahh} \\
\textbf{Haa} \\
\textbf{aefj aqf u} \\
\textbf{t41} \\
\textbf{Qunath} \\
\textbf{Hau}
For sphere $X_n = 2 = Z_n = 2 \neq 0$ (south and north poles)

The conclusion is, we cannot comb hair on the sphere without making the singularities (vertices).

Topological index for an insulator

The Berry curvatures and the Chern number.

For an insulator, we can use the Bloch waves to define a curvature in the 2D - k space. For Bloch waves $\Psi_{h,k}(\mathbf{r}) = U_{h,k}(\mathbf{r}) e^{i\mathbf{k} \cdot \mathbf{r}}$, we define the k-space curvature (Berry curvature) as

$$F_{h}(k) = \oint_{\text{unit cell}} |\nabla_{\mathbf{k}} U_{h,k}(\mathbf{r})|^2 \times \nabla_{\mathbf{k}} U_{h,k}(\mathbf{r}) \, d\mathbf{F}$$

where $\nabla_{\mathbf{k}}$ defines the vector in the k-space.

$$F_{h}(k) = \int_{\text{u.c.}} \left| \frac{\partial}{\partial \mathbf{k}} U_{h,k}(\mathbf{r}) \right|^2 \times \frac{\partial}{\partial \mathbf{k}} U_{h,k}(\mathbf{r}) \, d\mathbf{r}$$

$\epsilon_{ij}$ = Levi-Civita symbol, $\epsilon_{xx} = \epsilon_{yy} = 0$,

$\epsilon_{xy} = -\epsilon_{yx} = 1$

From the math proof, the Berry curvature and the Gaussian curvature are the same.
The total Berry curvature is the topological index!

The topological index is defined as

\[ C_n = \frac{1}{2\pi} \oint F_n(q) \, dq \equiv \text{the Chern number} \]

\[ B_2 \]

For each band \( n \), we can define such a number \( C_n \) and for an insulator the total Chern number:

\[ C = \sum \frac{C_n}{n} \]

over the filled bands.

- The total Chern number \( C \) is the same as the number of chiral edge states.
  e.g., if \( C = 0 \) we have a trivial insulator without edge states \( 6xx = 6xy = 0 \).

- If \( C \neq 0 \) we call such an insulator a \textbf{TI} or the Chern insulator.

  This insulator will have the edge states with \( 6xx = 0 \) and \( 6xy \neq 0 \). For the Hall conductivity:

\[ \sigma_{xy} = \frac{e^2}{h} \sum n_i \text{valence band} \left[ \frac{1}{2\pi} \oint F_i(q) \, dq \right] + \frac{e^2}{h} \sum n_i \text{conduction} \]
- Few important points. For the Gaussian curvature, the total $K$ is only quantized if the surface has no boundaries.

- For the Berry curvature is the same. If we integrate over the whole $BZ$, we will have a quantized Chern number. However, if we integrate over a part of $BZ$, the $C$ is now integer.

- For a metal we need to integrate only over the filled states or the Fermi sea, and as such there is a boundary set by the Fermi surface. As such $C_0$ is not quantized. That is why we have no quantized Hall conductivity for metals but we do have this for insulators.

- So by Chern $C$ we define TI but not Topological metals.

OTHER TOPOLOGICAL INDICES.

If in addition to the $C$ number, we demand a certain symmetry to be present e.g., time-reversal (TR) we can introduce different top indices.
If any of these indices are non-zero, the insulator is also a TI.

This kind of insulators also called the symmetry-protected insulators, with the common properties:

- One of the indices is non-zero.
- The bulk is an insulator, but the edge is a metallic state.
- The edge state is different from a simple metal in 3-1 dimensions, e.g. $\frac{1}{e}$ of the ordinary metal.
- The edge states may have some quantization effect.
- If the symmetry is broken, the edge state disappear.

Note: if we assume no symmetry the only TI is the Chern insulator, which is defined in the even space dimensions, e.g. we can have QHE only in 2D but not in 3D.

Note: for SPTIs they can exist for both 2D & 3D if we preserve TR symm. (e.g. NO MAGNETISM)

In 1D we need a very special symmetry called the chiral symmetry to get a TI.
Q. Why TI have metallic states at the edge?

Consider vacuum above the TI. Vacuum is an insulator (though trivial) with $C = 0$ inside the TI, $C \neq 0$. 

NB! Topology never changes in a smooth way! We cannot deform a sphere into a torus similarly, we cannot transform a trivial or a band insulator into a TI, thus the insulating states need to be destroyed by closing a band gap, or we get a metal.

Q. Why there is a metallic region between two plateaus?

6xy different plateaus have different topological indices $n$. So the story as above to go from $n=1 \rightarrow n=2$ need to close a gap to destroy the topology $\Rightarrow$ metal.

Q. Why the Hall conductivity is so exact in a Chern TI?
Since the Hall conductivity is determined by topology of the wave function, it is very robust and precise.

So, as long as any perturbation is not changing topology (or destroy symmetry), $\sigma_{xy}$ will be the same for any sample.

In order to do this, via some kind of perturbation we need to close a gap first (via doping for example) and only then we can change $\sigma_{xy}$.

So technically, the error bar in $\sigma_{xy}$ is 0 (well within how well we know $\pi$ and $e$).

Q. So far you talked about non-interacting $e$-? What if you turn e-e interactions?

For weakly interacting electrons the same connection between topology and Hall (the Berry connection) still remains. No proof here.
More about Berry phase

BP describes phase accumulation due to a motion of some complex vector around a close loop in the complex vector space. DESCRIPTIVE VERSION

For a specific example, let's consider a triatomic molecule.

\[ |U_1\rangle \quad |U_2\rangle \quad |U_3\rangle \quad |U_4\rangle \quad |U_5\rangle = |U_0\rangle \]

The BP is defined as

\[ \phi = -\text{Im } \ln \left[ \langle U_1 | U_2 | U_3 | U_4 \rangle \right] \]

For a complex vector \( z = 121 e^{i\theta} \), \( \text{Im } \ln z = \phi \).

Consider now our triatomic molecule

\[ |U_a\rangle = |U_d\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \quad |U_b\rangle = \frac{1}{\sqrt{2}} \left( e^{2\pi i/3} \right) \quad |U_c\rangle = \frac{1}{\sqrt{2}} \left( e^{4\pi i/3} \right) \]

then BP is given by:

\[ \phi = -\text{Im } \ln \left[ \langle U_a | U_b | U_c | U_d \rangle \right] = -\text{Im } \ln \left[ \left( \frac{e^{i\pi/3}}{2} \right)^3 \right] = -\pi \]

At least mathematically the BP is independent of individual phases of \( |U_j\rangle \). Let's introduce a new set of \( N \) states

\[ |\bar{U}_j\rangle = e^{-i\beta_j} |U_j\rangle \]

We can show that in this case the BP is unaffected as \( e^{i\beta_j} \) along the path will cancel out.

So we should say that BP is gauge invariant and as such perhaps describes some kind of physics.
THE CONCEPT OF A PARALLEL TRANSPORT

Suppose we have a chain of states \( |u_0> \rightarrow |u_1>, \ldots, |u_N> \) with no special phase relation. Let us define a new set, except of states \( |\tilde{u}_0> \rightarrow |\tilde{u}_1>, \ldots \)

**Continuous formulation of BP**

In this formulation we parametrize the path by a real variable \( \lambda \) such that \( |u_\lambda> \) traverses the path as \( \lambda \) evolves from 0 to 1, i.e. \( |u_{\lambda=0}> = |u_0> \) and \( |u_{\lambda}> \) is a smooth function of \( \lambda \). Let's try to derive an expression similar to the discrete version.

\[
\ln \langle u_\lambda | u_{\lambda+\Delta \lambda} \rangle = \ln \langle u_\lambda | (1 + d\lambda \frac{du}{d\lambda} + \ldots) \rangle \\
= \ln (1 + d\lambda \langle u_\lambda | d\lambda u_{\lambda} \rangle + \ldots) = d\lambda \langle u_\lambda | d\lambda u_{\lambda} \rangle + \ldots
\]

Then BP is:

\[ \phi = -\text{Im} \int \langle u_\lambda | i d\lambda u_{\lambda} \rangle d\lambda \]

\[ \text{Re} \langle u_\lambda | d\lambda u_{\lambda} \rangle = \langle u_\lambda | d\lambda u_{\lambda} \rangle + \langle d\lambda u_{\lambda} | u_{\lambda} \rangle = 0 \]

\[ \downarrow \langle u_\lambda | d\lambda u_{\lambda} \rangle \text{ is pure imaginary and} \]

\[ \phi = \oint \langle u_\lambda | i d\lambda u_{\lambda} \rangle d\lambda \]

**Berry connection or Berry potential**

\[ A(\lambda) = \langle u_{\lambda} | i d\lambda u_{\lambda} \rangle = -\text{Im} \langle u_{\lambda} | d\lambda u_{\lambda} \rangle \]

**In terms of \( A(\lambda) \):**

\[ \phi = \oint A(\lambda) d\lambda \]

**Q:** How Berry connection changes under gauge transformation?
\[ |\tilde{u}_\lambda\rangle = e^{-i\beta(\lambda)} |u_\lambda\rangle \]

\[ \widetilde{A}(\lambda) = \langle \tilde{u}_\lambda | \partial_\lambda \tilde{u}_\lambda \rangle = \langle u_\lambda | e^{-i\beta(\lambda)} \partial_\lambda e^{i\beta(\lambda)} | u_\lambda \rangle = \langle u_\lambda | \partial_\lambda | u_\lambda \rangle + \beta'(\lambda) \]

So \( B \) potential is not gauge invariant! and it changes as:
\[ \widetilde{A} = A + \frac{dB}{d\lambda} \]. But what about \( B \) potential?

\[ 1 |\tilde{u}_\lambda\rangle = 1 |u_\lambda\rangle \Rightarrow \beta(\lambda) = \beta(\lambda) + 2\pi m \quad m = 0, 1, \ldots \]

\[ \int_0^1 \frac{d\beta}{d\lambda} d\lambda = \beta(\lambda) - \beta(\lambda) = 2\pi m \quad \text{so for } \]
\[ \tilde{\phi} = \int \tilde{A}(\lambda) d\lambda = \int \left( A + \frac{d\beta}{d\lambda} \right) d\lambda = \phi + 2\pi m \]

So \( B \) potential is still gauge invariant!

You can think of \( B \) as the phase which still left over after moving in the loop.

**Example**

Let me illustrate this by considering a real physical problem.

Imagine we have an eigenvector which is a ground state of some \( H \). We can smoothly evolve the ground state by changing \( \lambda \), which in our case can be magnetic or electric fields \( E \).

The ground state is independent of \( |\lambda| \) but depends on \( \hat{S} \) operator.
So we can write instead $|\psi_n\rangle$ to emphasize that $|\psi\rangle$ depends on the direction of the magnetic field and not on its magnitude $|B|$.

Q: What's the BP of $|\psi_n\rangle$ as $\boldsymbol{n}$ carried around a loop in the magnetic field.

Let's try a simple discrete version:
1. $\hat{n} \rightarrow \hat{z} \rightarrow \hat{y} \rightarrow \hat{x} \rightarrow \hat{z}$.

So we are tracing one octant of the sphere.

$$\phi = - \text{Im} \ln \left[ \langle \uparrow z | \uparrow x \rangle \langle \uparrow x | \uparrow y \rangle \langle \uparrow y | \uparrow z \rangle \right]$$

What we remember from QM is that a spinor in arbitrary direction $\hat{n}$ is given by:

$$|\uparrow n\rangle = \begin{pmatrix} \cos \frac{\Theta}{2} \\ \sin \frac{\Theta}{2} e^{i\Phi} \end{pmatrix}$$

$$|\uparrow x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad |\uparrow y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad |\uparrow z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ 

Ignoring the normalization factors:

$$\phi = - \text{Im} \ln \left[ (i)(1+i)(1) \right] = -\pi/4$$

**Exercise:** Show that for $N$ spinors, taking $N$ equally spaced values from $0$ to $2\pi$ gives:

a) $\phi = -N \tan^{-1} \left[ \frac{\sin^2(\Theta/2) \sin(2\pi/N)}{\cos^2(\Theta/2) + \sin^2(\Theta/2) \cos(2\pi/N)} \right]$ 

b) find $\phi(\Theta)$ for $N \to \infty$

c) for $\Theta = 45^\circ$, compute numerically $N = 3, 4, 6, 12, \ldots$ and compare to $N \to \infty$

**THE END**
Topological Quantum Mechanics

Q. Why some insulators are interesting e.g. QHE and the other are trivial e.g. a piece of plastic?

Consider, QHE: QHE: So quantum is important.

(2) Transport is a notion of charge. In an charge is related to a conjugate variable of charge, i.e. $\text{lin } p$ is conjugate of $\psi$.

Invariant under $\phi \rightarrow \phi + \alpha \phi$ implies the conservation of charge. So perhaps phase of the wave function is the key. Focus on phase. First, is phase of any importance?

A: Relative yes, absolute no. i.e.:

$$\langle \Psi_1 | \Psi_2 \rangle = e^{i \phi} \langle \Psi_1 | \Psi_2 \rangle$$

However,

$$\langle \Psi_1 | \Psi_2 \rangle = \frac{1}{\sqrt{2}} | \Psi_1 \rangle + \frac{1}{\sqrt{2}} | \Psi_2 \rangle \Rightarrow$$

$$\langle \Psi_1 | \Psi_2 \rangle = \frac{\langle \Psi_1 | \Psi_1 \rangle + \langle \Psi_2 | \Psi_2 \rangle - i \phi}{\sqrt{2}}$$

The physical meaning of $\phi$ is in the interference.

$$\langle \Psi_1 | \Psi_2 \rangle = \cos \frac{\langle \Psi_1 | \Psi_2 \rangle}{2} + \sin \frac{\langle \Psi_1 | \Psi_2 \rangle}{2} e^{i \phi}$$

Common use of the phase is in condensed matter. Phase of Bloch waves: Consider a particle moving in a periodic potential.

$$[ - \frac{1}{2m} \nabla^2 + V(r) ] \Psi(r) = E \Psi(r) \quad \text{where} \quad V(r) = V(r + \mathbf{a})$$

The eigenfunctions of this equations:

$$\Psi_n, \kappa = U_n(r, \kappa) e^{i \kappa \cdot r} \quad \text{Bloch wave functions}$$
Now what is the phase of the Bloch wave?  
As usual in QM, doesn't matter, as it's invariant under global shift in the momentum space.

\[ \psi_{\mathbf{k}}(r) = U_{\mathbf{k}}(r) \; e^{i \mathbf{k} \cdot \mathbf{r}} \quad \psi'_{\mathbf{k}}(r) = e^{i \phi} U_{\mathbf{k}}(r) \; e^{i \mathbf{k} \cdot \mathbf{r}} \]
is the same.

What's interesting is it is also invariant under local shift:

\[ \psi_{\mathbf{k}}(r) = e^{i \phi(r)} U_{\mathbf{k}}(r) \; e^{i \mathbf{k} \cdot \mathbf{r}} \]

Which means that there is no mixing of waves with different \( \mathbf{k} \) as time goes on and on.

Let's focus on local phase symmetry:

Let's briefly review the concept of gauge in real space and then we will switch to the momentum space.

\[ i \frac{\partial}{\partial t} \psi(r,t) = -\frac{\hbar^2}{2m} \psi(r,t) + V(r) \psi(r,t) \]

\( \psi(r) \) will follow exactly the same equation as \( \psi'(r) \).

And we say the system is invariant under global phase shift, or global phase symmetry.

Remember, each symmetry implies the conservation of something.

E.g., \( x \rightarrow x + a \) implies \( \frac{\partial}{\partial x} \) conserved.

So here for U(1) phase symmetry we expect

\[ -e^{i\phi} \]
is conserved.

Now we switch to the local U(1) phase symmetry.

E.g. \( \psi(r) \) \( \psi(r,t) = e^{i \phi(r,t)} \psi(r,t) \)
is Sch. e.g., still invariant? NO.

\[ \frac{\partial}{\partial t} e^{i \phi} \neq e^{i \phi} \frac{\partial}{\partial t} \] and \( D e^{i \phi} \neq e^{i \phi} D \).

But can we make the equation invariant? Yes if we introduce some "charge particle" trick.
Let's recall the problem of motion of a charged particle $e$. We change its $\vec{p} \to \vec{p} + e\vec{A}/c$ and change $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - e\vec{A}/c$

\[
(i\hbar \frac{\partial}{\partial t} - e\vec{A}/c) \psi(r,t) = \left(\frac{i}{\hbar} \gamma^\dagger \gamma - \frac{e}{c} \vec{A}/c\right) \psi(r,t) + \nabla r \psi(r,t)
\]

\[
(i\hbar \frac{\partial}{\partial t} - e\vec{A} + \hbar \frac{\partial}{\partial r} \Phi(r,t)) \psi'(r,t) = \frac{i}{\hbar} \gamma^\dagger \gamma - \frac{e}{c} \vec{A}' - \hbar \frac{\partial}{\partial r} \Phi(r,t) \right) \psi(r,t) + \nabla r \psi(r,t)
\]

if we define

\[
\Phi' = \Phi - \frac{\hbar}{e} \frac{\partial}{\partial r} \Phi(r,t)
\]

and $A' = A + \frac{e}{\hbar} \frac{\partial}{\partial r} \Phi(r,t)$

we get:

\[
(i\hbar \frac{\partial}{\partial t} - e\vec{A}') \psi'(r,t) = \frac{i}{\hbar} \gamma^\dagger \gamma - \frac{e}{c} \vec{A}' \right) \psi(r,t) + \nabla r \psi(r,t)
\]

The change of $\Phi \to \Phi' = \Phi - \frac{\hbar}{e} \frac{\partial}{\partial r} \Phi(r,t)$ and $A \to A' = A + \nabla r \Phi(r,t)$

is called the gauge transformation, and it keeps physics $\vec{E}$ and $\vec{B}$ the same. More popular way to write it is to absorb $\frac{\hbar}{e}$ into $\Phi$:

\[
\Phi \to \Phi' = \Phi - \frac{\hbar}{e} \frac{\partial}{\partial r} \Phi(r,t) \quad A \to A' = A + \nabla r \Phi(r,t) \quad \psi(r,t) \to \psi' = \psi(r,t) e^{i\frac{\hbar}{e} \frac{\partial}{\partial r} \Phi(r,t)}
\]

The local phase invariance is related to the gauge field.

**Berry Connection and Berry Curvature**

Let's introduce the Berry connection

\[
\vec{A}_n = -i \langle U_n, k | \nabla_k | U_{n+1} \rangle
\]

As you can immediately see this is a gauge field in the momentum space; very similar to the vector potential.

$U_{n+1} \rightarrow e^{i\vec{A}_n(k)} U_{n+1}$ to make the Bloch states invariant we need to do this:

$A_n \rightarrow A_n' = -i \langle U_{n+1} | e^{i\vec{A}_n(k)} \nabla_k e^{-i\vec{A}_n(k)} | U_n \rangle$
\[ |U_{nk} \rangle \rightarrow e^{i \varphi_n(k)} |U_{nk} \rangle \]
\[ A_n \rightarrow A'_n = -i \langle U_{nk} | e^{-i \varphi_n(k)} \nabla_k e^{i \varphi_n(k)} | U_{nk} \rangle \]
\[ = -i \langle U_{nk} | \nabla_k | U_{nk} \rangle + \nabla_k \varphi_n(k) \langle U_{nk} | U_{nk} \rangle = A_n + \nabla_k \varphi_n(k) \]

So the Berry connection changes like a gage field in the \( k \)-space.

**Berry curvature**

Recall \( A \) is not observable as its value depends on the choice of a gage.

The quantity with a physical meaning is a curl of it, which is a magnetic field \( B \).

\[ F_n = \nabla_k x A_n = -i \varepsilon_{ijk} \frac{\partial}{\partial k_i} \langle U_{nk} | \partial_k | U_{nk} \rangle = -i \varepsilon_{ij} \langle \partial_k U_{nk} | \partial_k U_{nk} \rangle \]

This value of \( F_n \) is known as Berry curvature, which is observable.

**Position operator in lattice.**

Without lattice: \( p = -i \hbar \partial \) and \( r = i \hbar p \).

What about lattice?

One can prove that for the Bloch waves

\[ r = (2\pi)^2 \delta_{kn} - \Delta m \]

where \( m \) and \( n \) are the Band indices and

\[ \Delta m = -i \langle U_{nk} | \nabla_k | U_{nk} \rangle \]

Note if \( m = n \) it turns into the Berry connection.

If separation between bands is large we can zoom in to one band and ignore many other.

\[ p = -i \partial_k - \frac{e}{\hbar} A \]

For Bloch waves the conjugate require the gauge field. Berry connection is such a field.
Berry curvature and the Hall effect.

see Haldane, Phys. Rev. Lett. 93, 206602 (2001)

In the presence of \( E \) and \( B \)

\[
\frac{dp}{dt} = F = \dot{E} + ev \times B \\
\frac{dr}{dt} = \nabla_p \epsilon(\rho) + \frac{dp}{dt} \times [\nabla_p \times A(p)]
\]

Next I will replace \( p \) by \( k \).

\[
\frac{dr}{dt} = \frac{1}{\hbar} \left\{ \nabla_k \epsilon(k) + (\dot{E} + ev \times B) \times [\nabla_k \times A(k)] \right\} = \frac{1}{\hbar} \left[ \nabla_k \epsilon(k) + (\dot{E} + ev \times B) \times [\nabla_k \times A(k)] \right]
\]

Recall \( \nabla_k \times A(k) = F(k) \bar{e}_z \) while \( \bar{e}_z \) is along \( z \).

\[
\frac{dc}{dt} = \frac{1}{\hbar} \left\{ \nabla_k \epsilon(k) + (\dot{E} + ev \times B) \times [\nabla_k \times A(k)] \right\} = \frac{1}{\hbar} \left[ \nabla_k \epsilon(k) + (\dot{E} + ev \times B) \times [\nabla_k \times A(k)] \right]
\]

If all electrons have the same velocity: \( J = ev = \frac{eN}{A} \)

But in reality for particles with different \( k \) velocities are different so \( N \vec{v} = \sum_{n,k} \vec{v}_{n,k} \)

in all occupied states.

\[
\bar{J} = \frac{e}{\hbar} \sum_{n,k} \vec{v}_{n,k} = \frac{e}{\hbar} \sum_{n,k} \frac{dc}{dt} = \frac{e}{\hbar} \sum_{n,k} \frac{1}{\hbar} \left[ \nabla_k \epsilon_n(k) + (\dot{E} + ev \times B) \times [\nabla_k \times A(k)] \right] \]

The Hall effect comes from \( \dot{E} + ev \times B \) as it is the only term which generates \( \vec{J} \). And we can ignore all other terms, when we compute Hall conductivity. For Hall current \( \vec{J} \)

\[
\bar{J}_H = \frac{e}{A} \sum_{n,k} \frac{1}{\hbar} \dot{E} \times \bar{e}_z \bar{F}_n(k) = e \bar{e} \times \bar{e}_z \bar{A} \sum_{n,k} \frac{1}{\hbar} \bar{F}_n(k)
\]

For completely filled bands

\[
\bar{F}_n(k) = A \int \frac{d^2k}{(2\pi)^2} \bar{F}_n(k) \Rightarrow \bar{F}_n(k) = A \int \frac{d^2k}{(2\pi)^2} \bar{F}_n(k)
\]

\[
\bar{G}_{xy} = \bar{J}_H = \frac{e^2}{\hbar} \sum_{n,k} \bar{F}_n(k) = \frac{e^2}{\hbar} \left[ \sum \int \frac{d^2k}{(2\pi)^2} \bar{F}_n(k) \right] \text{ notice this is when all bands are fully occupied!}
\]
Dirac quantization, Gauss-Bonnet theorem
From math point of view $B$, Berry curvature $F_B$ and the Gaussian curvature $K$ are the same! So for simplicity we will use $B$ to calculate
- $\oint B \cdot ds$ and show it's quantized.
  $\oint B \cdot ds = \oint B \cdot ds = \frac{e}{2\pi} \cdot n$, where $n$ is integer magnetic charge which measures the # of \textit{magnetic monopoles inside} $M$
- $\oint k \cdot ds = 2\pi \chi_M$, where $\chi_M$ is even integer, known as the Euler characteristic, which measures the \textit{topological} nature of the manifold $M$

- $\oint g_z F \cdot dx = 2\pi e$ quantized, $\mathcal{C}$ is an integer called as the TKNN invariant or the Chern number

Magnetic Monopole and Dirac Quantization
(see M. Nakahara, Geometry, topology and physics, IOP)

For electric charge: $q e = \oint_B B \cdot ds$, $\nabla \cdot E = \rho$

For magnetic field: $\mu m = \oint_B B \cdot ds$, $\mu m = b/c \nabla \cdot B = 0$

But if there is a magnetic monopole:

$B = \mu m \frac{q}{r^2} = \mu m \frac{r}{r^2} = \frac{\mu m (x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$

$\nabla \times A = B$. The value of $A$ is not unique, but they are connected by a gauge transformation.

$A = \mu m \left(\frac{y_1, y_2, 0}{r(r-r_0)}\right) \Rightarrow \nabla \times A = \mu m \nabla \times \left(\frac{y_1-x_1, 0}{r(r-r_0)}\right) = \mu m \left(\frac{d_1 d_2 d_3}{r(r-r_0)}\right) x (\frac{y_1-x_1, 0}{r(r-r_0)}) = \mu m \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$

And it's singular at $r = r_0$.

In fact we can prove that, no matter what there will be always one singular point. The point is $A$ is NON singular but non observable.

Moreover for the gauge $\lambda = \mu m \frac{(-y_1, x, 0)}{r(r-r_0)}$ has a pole $r = -r$ (south pole)

$\begin{cases} 
\lambda_n = \mu m \frac{(-y_1, x, 0)}{r(r-r_0)} \\
\lambda_s = \mu m \frac{(y_1, -y_2, 0)}{r(r-r_0)}
\end{cases}$
At the equator the vector potential is multivalued, i.e. $A_N = A_S + 2\pi n \frac{(-y, x, 0)}{(r-z)(r+z)}$, at $z = 0$ $A_N = A_S + 2\pi n \frac{(-y, x, 0)}{r}$

$\phi \rightarrow \phi' = \phi - \frac{c}{\gamma t} A(t, r)$

$A \rightarrow A' = A + \frac{c}{\gamma t} A(t, r)$, $\psi(r, t) \rightarrow \psi'(t, r) = \psi(r, t) e^{i2\pi n}$
Here $A(r, t) = 2\pi n \gamma$

$\psi_N(r, t) = \psi_S(r, t) e^{i\frac{2\pi n q_e}{c \hbar} \gamma}$
also we know $\gamma$ and $\gamma + 2\pi$ are the same point.

$\gamma$: $\psi_N(r, t) = \psi_S(r, t) e^{i\frac{2\pi n q_e}{c \hbar} \gamma}$

and $\gamma + 2\pi$: $\psi_N(r, t) = \psi_S(r, t) e^{i\frac{2\pi n q_e}{c \hbar} (\gamma + 2\pi)}$

to have both equations valid $\frac{2\pi n q_e}{c \hbar} = \hbar$ $\psi_N = \psi_S e^{i\hbar n}$
$\psi_S = e^{i\hbar n}$$

Why charge is quantized?
Nobody knows.

Going back to the Hall conductivity

$\delta_{xy} = \frac{e^2}{h} \left[ \sum_{\text{all fully occupied}} \int_{\beta_2} d^2k \frac{f_n(k)}{2\pi} \right] = \frac{e^2}{h} \frac{\hbar}{c}$

END OF COURSE FOR 2018