What kinds of OP?

OP can be a scalar - periodic charge density in a crystal
vector - magnetization in a FM or FE
complex scalar - electron condensate wavefunction in SC
tensor - liquid crystals, anisotropic SC or superfluidity in He or high Tc SC.

Symmetries allowed in a physical system

Transitions

Ginzburg-Landau (SSB) with spontaneous symmetry breaking

even terms in F

2nd order

even + external fields

2 \rightarrow 1 : Weak 1st order

Topological without SSB

even + odd

1st order

\Phi or \tilde{r} dependent

periodically modulated structures

Complex, \chi dependent

Goldstone excitations

1st order

2nd order
• **Phase transitions** We saw that in Landau’s example, the parameter \( a \) in the free energy was temperature dependent. At a temperature \( T_c \), at which \( a \) changes sign, a phase transition takes place. The transition separates two distinct states of different symmetry. The low-temperature phase has lost some symmetry, more precisely it is missing a symmetry element.\(^5\)

• **New excitations** Our philosophy has been that every particle is an excitation of the vacuum of a system. When a symmetry is broken we end up with a new vacuum (e.g. a vacuum with \( M = -M_0 \)). The fact that the vacuum is different means that the particle spectrum should be expected to be different to that of the unbroken symmetry state (such as \( M = 0 \) in our example). We will see that new particles known as Goldstone modes can emerge upon symmetry breaking.\(^6\)

• **Rigidity** Any attempt to deform the field in the broken symmetry state results in new forces emerging. Examples of rigidity include phase stiffness in superconductors, spin stiffness in magnets and the mechanical strength of crystalline solids.

• **Defects** These result from the fact that the symmetry may be broken in different ways in different regions of the system, and are topological in nature. An example is a domain wall in a ferromagnet. These are described in Chapter 29.

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**Reading for this section:**

*QFT for CA* . Ch. 26
Recall our approach was to describe condensed matter from the excitation spectrum point of view. Why and very generally, we can ask what kind of excitations we can expect in a symmetry broken state.

Experimentally there are many examples of such excitations:
- isotropic ferromagnet with spin waves
- crystal with acoustic phonons
- etc.

Is there any general rule which tells if these excitation really exists?

Meet the Goldstone theorem:

If at the transition we break a continuous symmetry, there must exist in the ordered state of this material a collective mode or collective excitation with gapless energy spectrum.

But what about superconductivity?
In the electro-weak theory of Weinberg-Salam there is a combined $U(1) \times SU(2)$ gauge symmetry. Due to coupling to the Higgs field whose symmetry is spontaneously broken one gauge field remains massless (the photon) and the other three become massive. These massive particles are the $W^+$, $W^-$, and $Z$ bosons.

One of the key ideas first emphasized by Phil Anderson in 1963 was that a massless gauge field can acquire a mass in the presence of a coupling to a spontaneously broken field. A concrete realization of this occurs in superconductors. In the Meissner effect a superconductor thicker than the penetration depth expels magnetic fields. This is like the photon acquires a mass.


In a type II superconductor, vortices are allowed in the superconducting order parameter field. Can such vortices occur in the Higgs field? They may have been important in the early universe.

On fascinating thing is that for the Higgs field the crucial ratio [between the London penetration length and the superconducting coherence length] that determines whether type II behavior is possible is the ratio of Higgs boson mass to $W$ mass. The LHC results suggest that type II behavior is possible!

From P. Coleman's book, "Introduction to many-body..." page 246. Shortly after the importance of this mechanism for relativistic Yang Mills theories was noted by Higgs and Anderson, Weinberg and Salem independently applied the idea to develop the theory of "electro-weak" interactions. According to this picture, the universe we live is a kind of cosmological Meissner phase, formed in the early universe, which excludes the weak force by making the vector bosons which carry it, become massive. It is a remarkable thought that the very same mechanism that causes superconductors to levitate lies at the heart of the weak nuclear force responsible for nuclear fusion inside stars. In trying to discover the Higg's particle, physicists are in effect trying to probe the cosmic superconductor above its gap energy scale.

Vortices in a 200-nm-thick YBCO film imaged by scanning SQUID microscopy
Field theory and L-6 theory.

- Breaking symmetry with Lagrangian.

What we do in QFT is searching for a ground state of \( \psi(x) \).

For simplicity, let's start with a simple model:

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \psi \right)^2 - V(\psi) \quad \text{where} \quad V(\psi) = \frac{1}{2} \mu^2 \psi^2 + \frac{\lambda}{4!} \psi^4
\]

Now we move to a very interesting case: \( \mu^2 < 0 \).

In this case,

\[
\frac{\partial V}{\partial \psi} = 0 = -\mu^2 \psi + \frac{\lambda}{4!} \psi^3
\]

\[
\frac{\partial^2 V}{\partial \psi^2} = -\mu^2 + \frac{\lambda}{2} \psi^2
\]

\[
\implies \frac{\partial^2 V}{\partial \psi^2} = -\mu^2 + \frac{\lambda}{2} \psi^2 = 0 \quad \text{for} \quad \psi = 0
\]

This is very strange; our system has two new vacua, but it appears spontaneously.

What happens to excitations in the new ground state?

To investigate this lets select a new vacuum, e.g., \( \psi_0 \), and excite the field around the ground state. The Taylor expansion gives:

\[
V(\psi - \psi_0) = V(\psi_0) + \left( \frac{\partial V}{\partial \psi} \right)_{\psi_0} (\psi - \psi_0) + \frac{1}{2!} \frac{\partial^2 V}{\partial \psi^2} \bigg|_{\psi_0} (\psi - \psi_0)^2 + \ldots
\]

\[
= V(\psi_0) + \mu^2 (\psi - \psi_0)^2 + \ldots
\]

\[
\sim \psi^2
\]

The final Lagrangian is:

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \psi \right)^2 - \mu^2 \psi^2 + \frac{\lambda}{4!} \psi^4
\]

Let's compare this to the original theory:

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \psi \right)^2 - \frac{\mu^2}{2} \psi^2 + \frac{\lambda}{4!} \psi^4
\]

Notice, the Lagrangian doesn't break the symmetry, it is still \( \psi \rightarrow -\psi \) invariant but the symmetry is broken in the ground state. As a result, the vacuum gets a non-zero amplitude \( \psi_0 = \left( \frac{\mu^2}{\lambda} \right)^{1/2} \) and becomes heavier.
Goldstone Modes

Consider a 2-component QFT:

\[ \mathcal{L} = \frac{1}{2} \left[ (\partial \phi_1)^2 + (\partial \phi_2)^2 \right] + \frac{\mu^2}{2} (\phi_1^2 + \phi_2^2) - \frac{\lambda}{4!} (\phi_1^2 + \phi_2^2)^2 \]

It has SO(2) symmetry around internal \( \phi_1(x) - \phi_2(x) \)

There are infinite number of local minima.

\[ U(x) = -\frac{\mu^2}{2} x + \frac{x^2}{2} \frac{\lambda}{4!} \]

\[ x = \phi_1^2 + \phi_2^2 \Rightarrow \frac{dU}{dx} = 0 \Rightarrow \phi_1^2 + \phi_2^2 = \frac{6\mu^2}{\lambda} \]

Let's imagine we break symmetry e.g.

\[ (\phi_1, \phi_2) = (\pm \sqrt{6\mu^2/\lambda}, 0) \]

and investigate the excitations around the ground state.

\[ \phi_1' = \phi_1 - \sqrt{6\mu^2/\lambda} \quad \phi_2' = \phi_2 \]

See next figure to get a better idea.
Again as before we can check what happens if we consider our lagrangian for the new ound state.

Consider the lagrangian $U(\varphi_1, \varphi_2) = \frac{-\mu^2}{2}(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4!}(\varphi_1^2 + \varphi_2^2)^2$

We expand around minimum $(\varphi_1, \varphi_2) = \left( \sqrt{\frac{\mu^2}{\lambda}}, 0 \right)$

\[
\frac{\partial^2 U}{\partial \varphi_1^2} = 2\mu^2 \quad \frac{\partial^2 U}{\partial \varphi_2^2} = 0
\]

\[
\mathcal{L} = \frac{1}{2} \left[ \left( \partial_\mu \varphi_1 \right)^2 + \left( \partial_\mu \varphi_2 \right)^2 \right] - \mu^2 (\varphi_1')^2 + O(\varphi_1^4)
\]

So the particle with field $\varphi_1'$ has mass and $\varphi_2'$ is massless. See Fig. above

So the excitations in $\varphi_2'$ are gapless!

(in particle physics this would be massless)

The vanishing of the mass is the result of Goldstone theorem: 
[Breaking a continuous symmetry always results in massless excitations known as a Goldstone mode]
Goldstone theorem: Breaking a continuous symmetry always results in massless excitations known as a Goldstone mode.
The most amazing effect occur when we apply the same ideas to the broken ground state in a gauge theory.

Here we want to discuss the famous Higgs mechanism. Consider a scalar field theory:

\[
L = (\partial^\mu \psi^+ - ig A^\mu \psi^+)(\partial_\mu \psi + ig A^\mu \psi) + m^2 \psi^+ \psi - \lambda (\psi^+ \psi)^2 - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}
\]

This theory can describe for example an electron interacting with photons. The important point - this theory is gauge invariant, i.e.

\[
\left\{ \begin{array}{l}
A_\mu \rightarrow A_\mu - \frac{1}{2} \partial_\mu \alpha(x) \\
\psi \rightarrow \psi e^{i \alpha(x)}
\end{array} \right.
\]

The theory as written above describes

2 massive scalar particle \( E^p = (p^2 + m^2)^{1/2} \) and

2 transverse polarized massless photons \( E = p \)
Now we will break the symmetry. Let's move to the polar coordinates:
\[ \psi(x) = r(x) e^{i \theta(x)} \]
and select some unique angle \( \Theta \) for all \( x \).

So we start at this specific state and want to know what excitations can emerge around this state?

\[
\partial_{\mu} \psi + i g A_{\mu} \psi = \left( \partial_{\mu} r(x) \right) e^{i \theta(x)} + \\
i \left( \partial_{\mu} \theta(x) \right) r(x) e^{i \theta} + g A_{\mu} r(x) e^{i \theta} = \\
= \left( \partial_{\mu} r \right) e^{i \theta} + i r e^{i \theta} \left( \partial_{\mu} \theta + g A_{\mu} \right)
\]

Compare to \( \left( \partial_{\mu} \psi + i g A_{\mu} \psi \right) \)

we introduce a new gauge field \( A_{\mu} + \frac{1}{g} \partial_{\mu} \theta = C_{\mu} \).

So the term
\[
\left( \partial_{\mu} \psi^{\dagger} - i g A_{\mu} \psi^{\dagger} \right) \left( \partial_{\mu} \psi - i g A_{\mu} \psi \right) = \\
= \left( \partial_{\mu} r \right)^{2} + r^{2} \partial_{\mu} \theta^{2} - 2 g \psi^{\dagger} \psi C_{\mu} C_{\nu} \psi^{\dagger} \psi
\]

Show this
We can also transform the field energy $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$= \partial_\mu \phi - \partial_\nu \phi$$

Finally

$$\lambda = \left( \partial_\mu \phi \right)^2 + \phi^2 \left( \nabla^2 \phi \right) + \mu^2 \phi^2$$

$$- \frac{1}{4} F_{\mu \nu}^2 F_{\mu \nu} = \lambda$$

Now break the symmetry:

$$\rho = \sqrt{\mu^2 / 2 \lambda}, \quad \theta = 0$$

Let's introduce the excitations around the ground state:

$$\frac{\sqrt{2}}{\sqrt{2}} \rho = \lambda - \rho_0$$

Show this

$$L = \frac{1}{2} \left( \partial_\mu \chi \right)^2 - \mu^2 \chi^2 - \sqrt{\lambda} \chi \chi^3 - \frac{\lambda}{4} \chi^4$$

$$- \frac{1}{4} F_{\mu \nu} F_{\mu \nu} + \frac{m^2}{2} \phi \phi$$

$$+ \left( \frac{\mu^2}{\lambda} \right)^{1/2} \chi \phi \phi$$

$\frac{1}{2} \theta^2 c \phi c$
\[ \sqrt{\left( \frac{\mu}{x} \right) \sum} C_{\mu} C_{\mu}^* + \frac{1}{2} q^2 x^{2} C_{\mu} C_{\mu}^* + \ldots \]

\text{where } M = q \sqrt{\frac{m^2}{\Lambda}}
Now we see that we have field excitations with mass $\sqrt{2} \mu$

But $C^\mu$ which was massless gauge field analog of $A^\mu$ now has mass $M$.

Also $\Theta$ which was massless but broken symmetry now is not in the theory and we instead have a massive term $C^\mu$

The massless photon field $A^\mu$ has eaten $\Theta(x)$ and got mass $C^\mu(x)$ so we have:

\[ X \times 1 \quad E_p = \left( p^2 + (\sqrt{2} \mu)^2 \right)^{1/2} \]

+ 3 vector fields $C^\mu(x) \quad E_p = \left[ p^2 + \left( \frac{9^2 \mu^2}{\lambda} \right) \right]^{1/2}$

Summary: By applying a gauge transformation...
Summary: By applying a gauge transformation $A_\mu \rightarrow C \mu$ we remove massless Goldstone modes and gained mass in excitations combining it with the gauge field.