Fermi electrons in magnetic field
Let's submerge our electron into a solid and apply an external field \( B \).

The eq. of motion is given by:

\[
\frac{d\mathbf{p}}{dt} = -e \mathbf{v} \times \mathbf{B} \quad \Rightarrow \quad \text{the particle moves by spiral in FREE SPACE!}
\]

The condition that we can still use the quasi-classical approximation:

\[
\lambda \ll \frac{\hbar}{eB}
\]

\[
\lambda = \frac{2\pi}{k} = \frac{2\pi}{\mathbf{p}} \Rightarrow \frac{2\pi}{\mathbf{p}} \ll \frac{\mathbf{p}}{eB}
\]

\[
\frac{\hbar}{eB} \ll \frac{\mathbf{p}^2}{2\pi m_0}
\]

So, called the cyclotron frequency

\[
\omega_c = \frac{\hbar B}{m_0} \ll \frac{\mathbf{p}^2}{2\pi m_0}
\]

Now recall, \( \mathbf{p} \) inside the crystal cannot change unless we apply the external forces. So

\[
\frac{dp}{dt} = -e \mathbf{v} \times \mathbf{B} \quad \text{is still on if we assume} \quad \mathbf{p} \quad \text{is a quasi-momentum}
\]

\[
(\mathbf{v} \cdot \frac{dp}{dt}) = -e \mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0 \quad \text{and}
\]

\[
(\mathbf{B} \cdot \frac{dp}{dt}) = \ldots = 0
\]
\[ \left( \frac{dE}{dt} \right) = \text{conservation} \quad \text{every} \]

i.e., \( u = \frac{dE}{dp} \) and \( \left( \frac{dE}{dt} \right) = \left( \frac{dE}{dp} \right) \frac{dp}{dt} = \frac{dE}{dt} = 0 \)

Since \( \frac{dE}{dt} = 0 \), this means that

that the tip of the \( \vec{p} \) vector glides

on the surface \( E(\vec{p}) = \text{const} \)

From \( \left( \frac{dE}{dt} \right) = 0 \)

we get

\[
\frac{dp}{dt} = \frac{dp_{11}}{dt} + \frac{dp_{12}}{dt} \Rightarrow \frac{dp}{dt} = 0
\]

\[ \Rightarrow \text{a projection of the} \]

momentum

on the direction of \( \vec{B} \) is conserved

\[
\begin{cases}
p_{11} = \text{const} \\
E(p) = \text{const}
\end{cases}
\]

the common solution

gives the curve

which is the result of a cut of \( E(p) \)

by a plane which is \( \perp \) to the magnetic

field
Depending on topology of the F.S., we may end up with a closed or open trajectories. Here is the example of two types of trajectories:

Trajectory in real space

Statement: Trajectory of a quark particle in p-space defines its trajectory in r-space. To show this I will project p on the plane \( \perp \) to B.

\[
\frac{dp_\perp}{dt} = e \nu_\perp \times \vec{B} = e \frac{dr_\perp}{dt} \times \vec{B}, \quad \frac{dr_\perp}{dt} \propto \vec{B}
\]

\( \Rightarrow \) \( |p_\perp| = e |B| |r_\perp| \) \( \) 1) \( p_\perp \) scales with \( r_\perp \)

2) since \( \nu_\perp = \frac{dr_\perp}{dt} \) in \( r \)-space \( \perp \) to \( \frac{dp_\perp}{dt} \)

\( \) in \( p \)-space (\( \) from \( \frac{dp_\perp}{dt} = e \frac{dr_\perp}{dt} \times \vec{B} \))

This means each element of projection in \( r \)-space \( \perp \) each element in \( p \)-space, i.e., the trajectories are turned 90° wrt each other.
In short to see the trajectory in r-space rotate the plane by 90° and scale it by \( \frac{1}{eB} \) times. The direction of motion is the same.

Let's estimate a characteristic size of a trajectory in the xtal.

Recall \( \lambda_0 < \frac{PF}{eB} \Rightarrow P_F = \frac{\hbar}{\alpha} \), \( eB \approx \lambda_0 

then \( B < B_0 = \frac{\hbar}{e^2a^2} \approx 10^{-9} - 10^{-5} \) T

The largest ac field \( \approx 50T \)!

A condition for a cyclical motion in the field is \( \lambda > r = \frac{\hbar}{eB} \), or within each m.f.p. at least 1 turn must be completed. Now let's replace \( P_F \approx \hbar/a \); \( \lambda > \frac{\hbar}{e^2aB} \) \( \Rightarrow B > \frac{a}{e^2} \frac{\hbar}{\lambda^2} \approx \frac{a}{e} B_0 \approx \frac{a}{e^2} (10^4 - 10^5) \) T

since for pure metals \( \lambda \approx 10^3 - 10^5 \), \( a \Rightarrow \) The cyclical trajectory is found already at few Tesla!
Energy Spectrum of quasiparticles in Magnetic field

Ideal gas of electrons:

\[ E(\mathbf{p}) = \frac{p^2}{2m_0} \]  

Let us separate those

\[ E = \frac{p_x^2 + p_y^2}{2m_0} + \frac{p_z^2}{2m_0} = E_\perp + E_\parallel \]

Recall the density of states for 2D is \( \text{const}(E) \)

\[ \rho^{2D}(E) = \frac{m_e^*}{\pi \hbar^2} \]

Every energy level is degenerate for each \( E \) we have many \( p_x \) and \( p_y \) such as \( p_x^2 + p_y^2 = ? \) for many \( n_x \) and \( n_y = 2m_0 E_\perp \)

In the plane \( \perp B \) electrons move on the circle of \( r_c = \frac{m_0 \sqrt{v}}{e B} \) with \( \omega_c = \frac{eB}{m_0} \)

The energy is quantized:

\[ E = E_\perp + E_\parallel = \frac{\hbar \omega_c}{2}(n + \frac{1}{2}) + \frac{p^2}{2m_0} \]

\( n = 0, 1, 2, \ldots \)
For energy $E_1$ we have only energies $\frac{1}{\hbar} (n + \frac{1}{2}) \frac{1}{\hbar}$ separated by $\frac{1}{\hbar}\hbar$. For $n < \frac{1}{2}$ we get $E_{n} = \frac{p^2}{2m_0} = \frac{b^2}{2m_0} \left( \frac{2\pi n}{L_x} \right)^2 \hbar^2$

huge # of states almost quasicontinuous. See page 5.

For discrete values of $E_1$ in the quasiclassical approximation corresponds a specific trajectory, which depends on the quantum $n$. Then our condition

$\lambda_0 \ll \xi_E$ is equal $\hbar \omega c \ll E_f$
To find the radius $R_{B_n}$, let's compare the classical and quantum energies.

$$E_{\text{classical}} = \frac{\hbar^2}{m_0 w_c^2} R_{B_n}^2$$

and

$$E = \hbar w_c (n + \frac{1}{2})$$

From this we get

$$R_{B_n} = \sqrt{\frac{2 \hbar}{m_0 w_c} (n + \frac{1}{2})} = \sqrt{\frac{2 \hbar}{eB} (n + 1/2)}$$

So for the electron to go from the orbit $n$ to $n+1$ needs to get a nice $\hbar w_c$.

- For the same $n$, $E$ - get the same $R_{B_n}$, but $h_2$ and $p_2$ can be different.

**Let's include spin**

For electron with mag. moment $\mu_B = e \hbar / 2m_0 c$

its energy in $B = -\mu_B B$ i.e. with the spin we split a Landau level into 2 sub levels depending on $\mu \uparrow \downarrow B$ or $\mu \uparrow \downarrow -B$

$$E(n, s, k_z) = \frac{\hbar}{2} w_c (n + \frac{1}{2}) + s \mu_B B + \frac{\hbar^2 k_z^2}{2m_0}$$

$s = \pm 1$.

$s = +1 \Rightarrow ^{+}$ state & $s = -1 \Rightarrow ^{-}$ state is the lowest level $0$.

Note: Spin removes the Landau degeneracy for the same $n$ we have $n, s = +1$ and $s = -1$.

(see page 8)
Each state on the parabola is strongly degenerate.

Spin degeneracy is only absent for 0⁻.

Since E continuously depends only on \( p_z \), it looks like we have a quasi-1D system!

**Distribution of electrons in p-space**

Assume for have a single zone metal with a spherical Fermi surface.

If \( B = 0 \) all the states are inside the sphere and occupy \( (2\pi h)^3 \). So we mark the \( p_y \) points separated by \( 2\pi h \). The maximum circle is \( p_F = \sqrt{2m^*E_F} \). For any other \( X \)-section the states fill up the circle of \( \sqrt{p_F^2 - p_z^2} \); as \( p_z \to p_F \) the radius goes \( \to 0 \).
The uniform distribution of states with \( p_x, p_y, p_z \) corresponds to \( E = E(p_x, p_y, p_z) \) when \( 0 < |p| < p_F \).

Now we turn on \( B \); in the plane \( \hbar w_c (n + \frac{1}{2}) \) for \( p_z = \text{const} \), to find the radius \( p_n \)

we write down \( E_{\text{class}} = \frac{p_x^2 + p_y^2}{2m^*} = \frac{\hbar^2}{2m^*} \)

\( E_{\text{quantum}} = \frac{\hbar^2}{2\hbar w_c (n + \frac{1}{2})} \)

\[ \Rightarrow p_n = \sqrt{\frac{2m^* \hbar w_c (n + \frac{1}{2})}{\hbar^2}} \] (see fig 6 in page 8)

In other words: all states which we had confined inside the orbits with a radius \( p_n \), \( n = 0, 1, 2, \ldots \), now collapse on the circles; see fig. a vs. b in page 8.

Note the area in a) \( \pi \left(p_{n+1}^2 - p_n^2\right) = \)

Except for 0-state:

\( \pi p_0^2 = \pi m^* \hbar w_c \)

So for each allowed orbit we have the same # of \( E^- \) \( N_L = \frac{m^* l_x l_y \hbar w_c}{\pi \hbar^2} \) \( \Rightarrow \)

degeneracy of those \( p_n \) orbits is the same as the discrete Landau levels.

Note since \( p_n \) is independent of \( p_2 \) all...
Orbits are of the same radius then we deal with the Landau cylinders.

- Number of states filled up by \( e^- \) on each cylinder depends on its length within \( p_z \).

- With increasing \( p_z \) length ↓

- \# of cylinders ↓ with increasing ↓

To be cont'd