Function Spaces: Measure Theory

The most important first application of Hilbert space would be Fourier series. For this, we need careful definition of space of functions and the inner product on it. In turn we need a more general theory of integration via measure theory and Lebesgue integral, for that purpose.

Def.: Let \( \Sigma \) be a set. A \( \sigma \)-algebra \( \Sigma \subset 2^\Omega \) is a set of subsets of \( \Omega \) which are closed under complement and countable union.

In other words, if \( S \in \Sigma \) then \( 2^\Omega - S = S \in \Sigma \).

If \( \{ S_n \} \) is a sequence in \( \Sigma \) then \( \bigcup_{n=1}^\infty S_n \in \Sigma \).
Note that, since $\bigcap_n S_n = \left( \bigcup_n S_n \right)^c$, countable intersections also belong to $\Sigma$.

An important $\sigma$-algebra over a topological space is the Borel algebra, which is the smallest $\sigma$-algebra including the open sets.

For $\mathbb{R}$, we start with open intervals $(a, b)$ and can construct the other members of the $\sigma$-algebra by the process of taking unions and complements, and further unions etc.

Def: A measure $\mu$ is a non-negative function $\mu : \Sigma \to \mathbb{R}^+ = \{ x \in \mathbb{R} | x \geq 0 \}$, s.t.

- $\mu(\emptyset) = 0$
- If $\{ S_k \}$ is a countable $\sigma$-set of disjoint members of $\Sigma$, then
\[
\mu \left( \bigcup_{k=1}^{\infty} S_k \right) = \sum_{k=1}^{\infty} \mu(S_k)
\]

This property is called countable additivity.

**Ex 1:** Think of probability distributions.

**Ex 2:** The Lebesgue measure on \( \mathbb{R} \) is a special one which is defined on the Borel algebra on \( \mathbb{R} \) in the usual topology.

\[
m(a, b) = b - a.
\]

For \( \mathbb{R}^n \), one could take open boxes

\[B = (a_1, b_1) \times \ldots \times (a_n, b_n)\]

\[
m(B) = \prod_{k=1}^{n} (b_k - a_k)
\]

Note that a singleton \( \{x\} \) is a measurable set. If \( s_1, s_2 \) and \( s, s' \in \mathbb{R} \), then

\[
m(s_1) + m(s_2, s') = m(s_2) - D \geq m(s).\]

So \( m(s_1) \leq m(s) \leq m(s_2) \) for all \( k \). Hence \( m(\mathbb{R}^d) = 0 \).

So countable subsets of \( \mathbb{R} \) has measure zero.

For example, \( m(\mathbb{Q}) = 0 \).
Now we move to functions.

Def: $f: S \rightarrow X$, for $X$ with a topology and $S$ with a $\sigma$-algebra $\Sigma$, is called measurable if for any open set $U \subset X$, $f^{-1}(U)$ is in the $\sigma$-algebra, i.e., $f^{-1}(U) \in \Sigma$.

Note: We do not need to invoke a measure, yet.

We would often talk of function $f: [a, b] \rightarrow \mathbb{R}$ or $f: [a, b] \rightarrow \mathbb{C}$ with Borel algebra on $[a, b]$.

It turns out that for complex function, the real and the imaginary part has to be measurable in the real sense. Also, for a real function $f$ we could define $f(x) = \max\{f(x), 0\}$ and $f(x) = -\min\{f(x), 0\}$.

$f = f_+ - f_-$, with $f_+$ and $f_-$ both non-negative. If $f$ is measurable, so are $f_+$, $f_-$.
Lebesgue integral is defined for nonnegative real functions by defining "step function." In practice, we define simple functions. For a measurable set \( S \) define
\[
\chi_S(x) = \begin{cases} 
1 & \text{if } x \in S \\
0 & \text{if } x \not\in S
\end{cases}
\]
characteristic function of \( S \).

**Def:** A simple function \( f \) (step function in the book) is a linear combination \( \sum_{k=1}^{n} \alpha_k \chi_{S_k} = f \) for collection of measurable sets \( S_k \) and real numbers \( \alpha_k \).

**Define:** \( \int_S f \, d\mu = \sum_{k=1}^{n} \alpha_k \mu(S_k \cap A_S) \) for any measurable set \( S \) and a simple measurable function \( f \).

For nonnegative real functions, the Lebesgue integral over a measurable set \( S \) is defined by
\[
\int_S f \, d\mu = \sup \left\{ \int_S g \, d\mu \mid g \leq f, g \text{ is simple and measurable} \right\}
\]

**Def:** For general real functions, \( \int_S f \, d\mu \) is defined \( \int_S f^+ \, d\mu \) and \( \int_S f^- \, d\mu \) are defined and at least one of the integrals are finite.

\[
\int_S f \, d\mu = \int_S f^+ \, d\mu - \int_S f^- \, d\mu
\]
Complex integrals are defined if integrals are defined.

After this long delay, we come to $L^2$ spaces. This space is made of measurable functions $f: \mathbb{R} \to \mathbb{C}$ so that

$$\int_{\mathbb{R}} |f|^2 \, dm$$

is defined and is finite.

It is a normed space $\|f\| = \sqrt{\int_{\mathbb{R}} |f|^2 \, dm}$ and it has a scalar product $(f, g) = \int_{\mathbb{R}} f \overline{g} \, dm$.

Note that $\int_{\mathbb{R}} f^2 \, dm = 0$ does not mean $f = 0$, but that $f \neq 0$ on a set of measure zero.

So, instead of dealing with $f$ we need to deal with equivalence classes of $f$ that differ only on sets of measure zero. We will use $[f]$ to indicate the equivalence class.

$L^2(\mathbb{R})$ is complete according to Riesz-Fischer theorem.

Thus: If $f_1, f_2, \ldots$ is a Cauchy sequence in $L^2(\mathbb{R})$, then it converges to an $f \in L^2(\mathbb{R})$. 

Signed

Date

Signed

Date
Fourier Series

We consider function $f: \mathbb{R} \to \mathbb{C}$ where $f(t + T) = f(t)$. We could focus on the primary domain to be $[-T/2, T/2]$.

And take $\phi_n(t) = \frac{1}{\sqrt{T}} e^{in\omega t}$ with $\omega = \frac{2\pi}{T}$ and $n \in \mathbb{Z}$.

Then $\phi_n(t)$ form an orthogonal system on $[-T/2, T/2]$.

$$\int_{-T/2}^{T/2} \phi_n(t) \phi_m(t) \, dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(n-m)\omega t} \, dt$$

$$= \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

If turns out that this is a complete orthogonal system, and therefore...
any \( f \in L^2 \left( \left[ \frac{1}{2}, \frac{3}{2} \right] \right) \) could be expanded as

\[
f = \sum_{n=-\infty}^{\infty} c_n \phi_n
\]

with

\[
c_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_n(t) f(t) \, dt
\]

To prove this one needs to know that trigonometric polynomials are everywhere dense in \( L^2 \). Textbooks often do this in two steps: First show a version Weierstrass approximation theorem, proving that continuous functions are well approximated by trigonometric polynomials. Then they show that \( L^2 \) functions are well approximated by continuous functions in the \( L^2 \) sense.
To get a sense how this might work, we will try a direct approach, without proof. Consider the function:

\[ K(\frac{t-u}{\sqrt{n}}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{t-u}{\sqrt{n}} \right)^2} \]

This function is very peaked around \( t = u \), for large \( n \) is chosen so that \( \int_{-\infty}^{\infty} K(\frac{t-u}{\sqrt{n}}) du = 1 \)

When \( n \) is large, \( K(\frac{t-u}{\sqrt{n}}) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{t-u}{\sqrt{n}} \right)^2} \)

for \( \frac{|t-u|}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \)

And \( \sqrt{n} \approx \frac{1}{\sqrt{n}} \frac{\sqrt{n} \pi}{1} = \frac{\sqrt{n} \pi}{2} \)

Now \( f(t) \) and \( \hat{f}(t) = \int_{-\infty}^{\infty} K(\frac{t-u}{\sqrt{n}}) f(u) du \) could be compared. Note that \( \hat{f}(t) \) could be written in terms of trigonometric polynomials. \( \hat{f}(t) \) is a slightly smoothed version of \( f \). Of course showing that they are close in \( L^2 \) sense requires more technical work.
Now we spend some time thinking about printing convergence of Fourier series.

For the sake of convenience, let us set

\[ T = 2\pi \]

\[ \phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{in x} \]

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n \phi_n(x) \]

\[ \langle n, f \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x) \, dx \]

Consider the partial sum

\[ f_N(x) = \sum_{n=-N}^{N} c_n e^{in x} \]

\[ \sum_{n=-N}^{N} e^{in x} = \frac{e^{iN^2} - 1}{e^{ix} - 1} \]

\[ \sum_{n=-N}^{N} e^{iNn} u = e^{-iNu} + e^{-i(N-1)u} + \ldots + e^{-i2u} + e^{-iu} \]

\[ = e^{-iNu} \left( 1 - e^{i2u} + \ldots + e^{i(N-1)u} \right) \]

\[ = e^{-iNu} \left( 1 - e^{i2u} + \ldots + e^{i(N-1)u} \right) \]

\[ = \frac{e^{-iNu} \left( e^{i(N+1)u} - 1 \right)}{1 - e^{i2u}} \]

\[ \frac{\sin \left( N + \frac{1}{2} \right) u}{\sin \frac{u}{2}} = D_N(u) \]
$D_N(u)$ is the Dirichlet kernel.

$$\int_{-\pi}^{\pi} D_N(u) \, du = \sum_{n=-N}^{N} \int_{-\pi}^{\pi} e^{inu} \, du = \frac{2\pi}{2\pi} \sum_{n=-N}^{N} \delta_{n,0} = 2\pi$$

$$S_0 f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x-y) f(y) \, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(u) f(x+u) \, du$$

We used $D_N(u) = D_N(u)$

Use transform $u \rightarrow u = x + u$

If $f$ is piecewise continuous and have left and right limits of $f(x)$ at $x_0$

$$f_N(x_0) = \frac{1}{2} \left( f(x_0) + f(x_0) \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(u) \left( f(x_0 + u) - f(x_0) \right) \, du + \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(u) \left( f(x_0 + u) - f(x_0) \right) \, du$$

These two integrals could be rewritten

$$\Delta^N = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( f(x_0 + 2\pi) - f(x_0) \right) \sin(N+1)u \, du$$

Continued on Page

If $f(x_0 + 2\pi) - f(x_0)$ is a smooth function
This integral vanishes as \( N \to \infty \).

If the functions are square integrable, the integrals are Fourier coefficients which would go to zero as \( N \to \infty \).

We can write in terms of real functions and use the orthonormal basis:

\[
\frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \ldots
\]

\[
f(x) = \frac{a_0}{\sqrt{\pi}} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) \]

\[
Example:\quad f(x) = |x|
\]

\[
f(x) = \frac{a_0}{\sqrt{\pi}} + \sum_{m=1}^{\infty} \frac{2}{\sqrt{\pi} m} \cos mx + \sum_{n=1}^{\infty} \frac{2}{\sqrt{\pi} n} \sin nx
\]

\[
a_0 = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} |x| \, dx = \frac{2}{\sqrt{\pi}} \int_{0}^{\pi} x \, dx = \frac{\pi^2}{6}
\]

\[
a_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\sqrt{\pi}} \int_{0}^{\pi} x \cos nx \, dx
\]

Let us do it this way:

\[
\int_{-\pi}^{\pi} \sin nx \, dx = \left[ \frac{-\cos nx}{n} \right]_{-\pi}^{\pi} = \frac{\cos n\pi - \cos (-n\pi)}{n}\]

Continued on Page

Read and Understood By
Differentiable in $\mu$:
\[ \int_0^\pi \cos(\mu x) dx = \frac{1 - \cos \mu}{\mu^2} + \frac{\pi \sin \mu}{\mu} = \frac{\pi \sin \mu - (1 - \cos \mu)}{\mu} \]

Now set $\mu = 1, 2, 3, \ldots$

If $\mu = 2n$, $n = 1, 2, \ldots$,
\[ \sin 2\pi n = 0 \quad \text{and} \quad 1 - \cos 2\pi n = 0 \]

So $\mu = 2n + 1, n = 0, 2, 4, \ldots$

\[ \int_0^\pi (\cos(\mu x))^2 dx = \frac{\pi (2n+1) \sin(2n+1) \pi - (1 - \cos (2n+1) \pi)}{(2n+1)^2} = -\frac{2}{(2n+1)^2} \]

\[ f(x) = \frac{1}{\sqrt{\pi}} \frac{x^2}{\pi} + \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{2}{\pi} \left[ \frac{2}{(2n+1)^2} \right] \cos((2n+1)x) \]

\[ = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2} \]

Since the $n$-th term goes as $\frac{1}{n^2}$, this is an absolutely convergent series for all $x$. 

Read and Understood By

Signed ___________________________ Date ___________________________

Signed ___________________________ Date ___________________________
Now consider the discontinuous function \( f(x) = x \) for \( x \neq \pi n \)

We only get contribution from \( \sin nx \) line:

\[
f(x) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} b_n \sin nx
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx
\]

Consider the integrals:

\[
\int_{-\pi}^{\pi} \cos(\pi x) \, dx = \frac{2 \sin(\pi n)}{\pi}
\]

Differentiating in \( n \) we get:

\[
\int_{-\pi}^{\pi} x \sin(\pi x) \, dx = \frac{2 \sin(\pi n)}{\pi} + \frac{2 \pi \cos(\pi n)}{\pi^2}
\]

Now put \( n = 1, 2, 3, \ldots \):

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx = \frac{1}{\pi} \frac{2 \pi (-1)^{n-1}}{n!}
\]

\[
f(x) = \frac{1}{\sqrt{\pi \sqrt{\pi}}} \sum_{n=1}^{\infty} \frac{2 \pi (-1)^{n-1}}{n!} \sin nx = \sum_{k=1}^{\infty} \frac{a_k}{n^k}
\]

This series is not absolutely convergent at \( x \neq 0, \pi \).
Notice that at \( N \), the series gives 0, the average of left and right limits.

If one looks carefully at partial sums near the discontinuity, one sees jumps.

These jumps stay as \( N \to \infty \), but happen closer and closer to the discontinuity.

This is called the Gibbs phenomenon.

To see how it arises, consider the simple case

\[
 f(x) = 0 \text{ for } x > 0 \quad \text{and} \quad f(x) = -a \text{ for } x < 0
\]

For small \( x \)

\[
 f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(N+\frac{1}{2})u \frac{f(x+u)}{\sin u} \, du
\]

\[
 = -\frac{a}{2\pi} \int_{-\pi}^{\pi} \frac{\sin u}{u} \, du
\]

\[
 f_N(0) = -\frac{a}{2\pi} \pi = -\frac{a}{2}
\]

we get \( \pi \) as expected.

What if we put \( x = \frac{2\pi}{2N+1} \)?

\[
 \int_{-\pi}^{\pi} \sin(N+\frac{1}{2})u \frac{1}{\sin u} \, du
\]

change var to \( \theta = \frac{\sin u}{2} \)

\[
 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{\sin \frac{\theta}{2}} \frac{\sin \frac{\pi}{2} - \sin \frac{5\theta}{2}}{\sin \frac{\theta}{2}} \, d\theta
\]

we get a negative answer.

Read and Understood By

\[
 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} \, d\theta = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \frac{3\theta}{2} \, d\theta = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin \frac{3\theta}{2}}{\pi} \, d\theta
\]

Signed Date

\[ \text{jump more past, better } N \]
Fourier Transform

Let us bring back $T$

\[ f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} C_n e^{-\frac{2\pi i n t}{T}} \quad \text{for } t \in [-\frac{T}{2}, \frac{T}{2}] \]

\[ C_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{\frac{2\pi i n t}{T}} f(t) \, dt \]

Call $\omega_n = \frac{2\pi n}{T}$, $\Delta \omega = \frac{2\pi}{T}$

\[ f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} C(\omega_n) e^{-i\omega_n t} \Delta \omega \]

\[ C(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t) \, dt \]

The function $C$ is the Fourier transform of the function $f$.

If we let $T \to \infty$

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(\omega) e^{-i\omega t} \, d\omega \]

\[ C(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t) \, dt \]
\[ \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{T} \sum_{n} \left| c_n \right|^2 \]  

\[ \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{T} \sum_{n} \left| c_n \right|^2 = \frac{1}{2\pi} \sum_{n} \left| c_n \right|^2 \Delta \omega \]

When \( T \to \infty \)

\[ \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |C(\omega)|^2 \, d\omega \]

This is the Plancherel's formula.

**Example:** \( f(t) = Ae^{-\alpha t^2} \)

\[ C(\omega) = A \frac{e^{-\alpha \omega^2}}{\omega^2 + \alpha^2} \]

Read and Understood By

<table>
<thead>
<tr>
<th>Signed</th>
<th>Date</th>
<th>Signed</th>
<th>Date</th>
</tr>
</thead>
</table>
**Convolution Theorem**

**Def:** The convolution of $f$ and $g$, is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-u) g(u) \, du$$

Let $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k)e^{-iwt} \, dk$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d(\omega)e^{-iwt} \, d\omega$$

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(k)e^{-iwt} d(\omega)e^{i\omega t} \, d\omega \, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(k)d(\omega)e^{-i(\omega t - \omega u)} \, d\omega \, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(k)d(\omega)e^{i\omega u} \, d\omega \, dk$$

So $f * g$ has the Fourier transform which is the product of the Fourier transforms of $c * d$. 

Read and Understood By

<table>
<thead>
<tr>
<th>Signed</th>
<th>Date</th>
<th>Signed</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Multidimensional Fourier transform

\[ f(\mathbf{k}) = \frac{1}{(2\pi)^n} \int e^{i \mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}) \, d^n \mathbf{x} \]

\[ \tilde{f}(\mathbf{k}) = \frac{1}{(2\pi)^n} \int e^{-i \mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}) \, d^n \mathbf{x} \]

That \( f(\mathbf{x}) = \frac{1}{(2\pi)^n} \int e^{i \mathbf{k} \cdot \mathbf{x}} \left( e^{-i \mathbf{k} \cdot \mathbf{y}} \tilde{f}(\mathbf{y}) \right) \, d^n \mathbf{y} \)

Formally,

\[ \mathcal{S} = \sum_{\mathbf{k}} \int e^{i \mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \tilde{f}(\mathbf{y}) \, d^n \mathbf{y} \]

This is formally captured by

\[ \frac{1}{(2\pi)^n} \int e^{i \mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} d^n k = \delta(\mathbf{x} - \mathbf{y}) \]