Differential Equation - Analytic Methods

In this section we will take a look at differential equations, taking advantage of the possible use of the tools we learnt about (complex) analytic functions.

General system of first order equations

\[ \frac{d}{dz} u_k(z) = h_k(u_1, \ldots, u_n, z) \quad k = 1, \ldots, n \]

Note that a general n-th order diff. eq.

can be recast this way, at least locally:

\[ F(u, u', \ldots, u^{(n)}, z) = 0 \]

Solve this to rewrite

\[ u^{(n)} = f(u, u', \ldots, u^{(n-1)}, z) \]

Now define \( u_k = u^{(k-1)} \), namely \( u_1 = u, u_2 = u', \ldots, u_n = u^{n-1} \)

\[ u_1' = u_2 \]
\[ u_2' = u_3 \]
\[ u_n' = u_n \]
\[ u_n' = f(u_1, \ldots, u_n, z) \]
Example 1: \( m \frac{d^2x}{dt^2} = -V'(x) \) 3. One second-order equation

\( U_1 = x \quad U_2 = \frac{dx}{dt} \)

\( \frac{dU_1}{dt} = U_2 \quad \frac{dU_2}{dt} = \frac{1}{m} V'(x) \) 3. Two first-order equations.

For the general first-order equation we need \( u(t_0) = u(x_0) \), the "initial conditions" to determine a solution. Intuition:

\[
\begin{align*}
\delta & = \delta x = \delta \bar{y} \\
\delta z & = \delta \bar{z} \\
\delta x & = \delta \bar{y} \\
\delta y & = \delta \bar{z} \\
\end{align*}
\]

Can this intuition be made to work analytically? A mostly

For complex analytic differential solution may be path dependent

and depend on analytic continuation.
For an autonomous system

\[ \frac{du_k}{dt} = h_k(u_1, \ldots, u_n) \quad k = 1, \ldots, n \]

one can make it locally into an \( n-1 \) variable non-autonomous system.

Call \( u_n = t \)

\[ \frac{du_k}{dt} = \frac{h_k(u_1, \ldots, u_n, t)}{h_n(u_1, \ldots, u_{n-1}, t)} \quad n = 1, \ldots, n-1 \]

Example:

\[ m \ddot{x} = -kx \]

\[ \Rightarrow \begin{align*}
  \frac{du_1}{dt} &= u_2 \\
  \frac{du_2}{dt} &= -\frac{k}{m} u_1
\end{align*} \]

\( u_1 = x \)
\( u_2 = \dot{x} \)

Note that \( \frac{du_2}{du_1} = -\frac{m}{k} \frac{u_2}{u_1} \)

Use variable separable method \( \int u_2 du_1 = -\frac{k}{k} \int \dot{x} \ddot{x} \)

\[ m \left( \frac{dx}{dt} \right)^2 + \frac{k}{2} x^2 = \text{const} \]

\[ \left[ \frac{m}{2} \left( \frac{dx}{dt} \right)^2 + \frac{k}{2} x^2 = \text{energy} \right] \]
Scale invariant equation:

\[ z \rightarrow a^z \quad \text{an invariance} \]

**Example:**

\[ \frac{du}{d\tau} = \frac{1}{2} g(u) \]

**Trick:**

\[ z \frac{du}{d\tau} = g(u) \Rightarrow \frac{du}{d\tau} = g(u) \quad \text{where } z = e^\tau \]

\[ = D \int \frac{du}{g(u)} = z + c \Rightarrow z = C e^{\int \frac{du}{g(u)}} \]

For \( g(u) = e^u \)

\[ z = C e^{\int \frac{du}{g(u)}} \Rightarrow \left( \frac{z}{C} \right) = \left( \frac{u}{u_0} \right)^\tau \]

\[ = D \quad u = u_0 \left( \frac{z}{C} \right)^\tau \]

**Isobaric ODE:**

\[ z \rightarrow a z \quad u \rightarrow a^u \]

(scale covariant)

Try \( u = z^p v \). The function is invariant \( \Rightarrow D \) scale invariant equation.

**Example:**

\[ \frac{du}{d\tau} = \frac{u}{z} - z \]

\[ u = a^z, \quad z = a^z \quad \text{is an invariance} \]

**Introduce** \( v \) s.t. \( u = z^p v \)

**Note** \( u \) invariant under transformation.

\[ \frac{z}{z^2} \frac{dv}{dz} + 2z \frac{dv}{dz} = 2v - 2 \stackrel{?}{\text{Get rid of extra factor of }} dz\]

\[ z dv = -v - 1 \]

Now ready to use tricks from scale invariant equation.

\[ \ln z = -\ln (z^2 + 1) + \text{const} \]

\[ v = \frac{C + 2}{z} \]

\[ = D \quad \ln z = \ln \left( \frac{z}{z^2} \right) \]

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Read and Understood By

\[ z = \frac{C + 2}{z} \quad \Rightarrow DU = z(A - z) \]

Signed Date

\[ z = \frac{C + 2}{z} \quad \Rightarrow DU = z(A - z) \]

Signed Date
First Order Diff Eqs

Linear first order eqs

Homogeneous Linear Eq

\[ u'(z) + p(z) u(z) = 0 \]

\[ \frac{du}{u} = -p \quad \Rightarrow \quad u(z) = u(z_0) e^{-\int_{z_0}^{z} p(\zeta) d\zeta} \]

By Th., notice that if \( p(z) \) has a simple pole at \( z_1 \), then

\[ \int_{z_0}^{z} p(\zeta) d\zeta = \alpha \ln (z_2 - z_1) + \cdots \]

and \( u(z) \sim (z - z_1)^\alpha \), branch pt. if \( \alpha \notin \mathbb{Z} \)

So meromorphic "coefficients" could give rise to solutions with branch points

Now we can solve the inhomogeneous problem

\[ u'(z) + p(z) u(z) = f(z) \]

Define \( h(z) = e^{-\int_{z_0}^{z} p(\zeta) d\zeta} \)

and let \( u(z) = h(z) w(z) \)

\[ h(z) w'(z) = f(z) \quad \Rightarrow \quad w(z) = u(z_0) + \int_{z_0}^{z} \frac{f(\zeta)}{h(\zeta)} d\zeta \]

\[ \Rightarrow \quad u(z) = (u(z_0) + \int_{z_0}^{z} \frac{f(\zeta)}{h(\zeta)} d\zeta) \frac{1}{h(z)} \]
This is just the method of integrating factor.

Riccati equation

We will mention a particular kind of first order nonlinear differential equation, namely, Ricatti eqn.
\[ u' = q(x) + 2q(x)u + 2x(x)u^2 \]

It is interesting that this can be transformed into a linear second order equation.

We have seen such a thing in the context of quantum mechanics.
\[ \psi(x) e^{ix'x} - \frac{i2}{2m} \psi''(x) + V_0 \psi = E \chi \]
\[ D^2 = \frac{i2}{2m} \left( \frac{\partial^2}{\partial x^2} + i \frac{\partial}{\partial x} \right) \psi(x) + V_0 \psi(x) - E = 0 \]

If \[ \frac{\partial}{\partial x} \]
\[ \psi' = \chi \]
\[ \psi'' = \frac{2m}{i} \left( E - V_0 \right) \chi - \frac{i}{2m} \chi^2 \]

This is the beginning of the WKB approximations.

We thus have \[ \psi(x) = e^{-\frac{i}{2m} \int (E - V_0(x)) dx} \]
\[ \frac{y'(z)}{y(z)} = -\frac{1}{q(z)} \frac{y''(z)}{y(z)} \]

\[ u'(z) = -\frac{q(z)}{q_2(z)} \frac{y'(z)}{y(z)} + \frac{1}{q_2(z)} \left( \frac{y'(z)}{y(z)} \right)^2 - \frac{1}{q_2(z)} \frac{y''(z)}{y(z)} \]

\[ q_0(z) + q_1(z) u(z) + q_2(z) u(z)^2 = \frac{q_0(z)}{q_2(z)} - \frac{q_1(z)}{q_2(z)} \frac{y'(z)}{y(z)} + \frac{q_2(z)}{q_2(z)} \frac{y''(z)}{y(z)} \]

So Riccati eqn \( \Rightarrow y''(z) = \left( \frac{q_1(z)}{q_2(z)} \right) y'(z) + q_2(z) y(z) \)

which is a second order linear differential equation.

We will discuss such equations soon.

Interestingly, if we know one particular solution of the Riccati equation of the form \( u_0(z) \), we can try to find a class of solutions as follows:

Let \( u(z) = u_0(z) - \frac{c}{u(z)} \), while \( u'_0(z) = q_0(z) u_0(z) + q_1(z) u_0(z)^2 \)

\[ u'(z) = u'_0(z) - \frac{c}{u_0(z)} u'(z) \]

\[ q_0(z) + q_1(z) \left( u_0(z) - \frac{c}{u_0(z)} \right) + q_2(z) \left( u_0(z)^2 - \frac{c}{u_0(z)} \frac{2c}{u_0(z)} \right) \]

\[ = q_0(z) + q_1(z) u_0(z) + q_2(z) u_0(z)^2 - \frac{c}{u_0(z)} \left( q_1(z) 2u_0(z) + 2q_2(z) u_0(z)^2 \right) - cq_2(z)^2 \]

\[ \Rightarrow u'(z) = \left( q_1(z) + 2q_2(z) u_0(z) \right) u_0(z) - cq_2(z)^2 \]

which can be solved by the method of integrating factor.
General first-order differential equation, exact differentials and integrating factors.

The general first-order equation is of the form
\[ g(u, z) \, du + h(u, z) \, dz = 0. \]

If \( g(u, z) = \frac{\partial F}{\partial u} \) and \( h(u, z) = \frac{\partial F}{\partial z} \)

then the equation is just \( \partial F = 0 \)
\[ \Rightarrow F(u, z) = \text{constant} \text{ in the solution}. \]

If the one form is \( df \), it is exact.

If the one form is not exact one may find \( \lambda(u, z) \) so that \( \partial \lambda(u, z)(g(u, z) \, du + h(u, z) \, dz) \)
is an exact form. Such a \( \lambda \) is called an integrating factor. Its existence can be proved in many cases but finding it practically may be hard.

[what we really mean is that we have a curve \( x(t) \mapsto (u(x(t)), z(x(t))) \), so that
\( (gdu + hdt)(dx) = 0 \). Such a curve would be a parametrized solution. The domain of \( t \) could be real intervals or complex regions depending on the context.]
Linear Differential Equations

Linear systems with constant coefficients

\[ \frac{d U(t)}{dt} = \sum_{k=1}^{n} A_{kk} U(t) \quad A_{kk}, k=1, \ldots, n \]

\[ \frac{d^2 U(t)}{dt^2} = A U(t) \]

\[ U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix} \quad A = \begin{bmatrix} A_{11} & -A_{1n} \\ -A_{n1} & A_{nn} \end{bmatrix} \]

Note that \( U(t) = e^{tA} u(0) \) solves this equation.

For the sake of convenience, set \( z_0 = 0 \).

Choose a basis in which \( A = D + N \) is diagonal.

In particular, choose the Jordan normal form

\[ A \rightarrow \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & \cdots & \lambda_n \end{pmatrix} \]
Let \( U(\lambda) = \sum \frac{\xi(\lambda)}{\lambda - \lambda_k} \).

\( \xi_1, \ldots, \xi_n \) being the basis vectors.

Note that \( \xi_1(\lambda) \)'s corresponding to the same Jordan block affect each other.

If each block is 1-dimensional (the case where \( A \) is diagonalizable), life is simple.

\[
\frac{d}{d \lambda} \xi_k(\lambda) = \lambda_k \xi_k(\lambda) \quad \Rightarrow \quad \xi_k(\lambda) = e^{\lambda_k \lambda} \xi_k(\lambda_0)
\]

What if there is a nilpotent component?

Let us look at the \( 2 \times 2 \) block case.

\[
\frac{d}{d \lambda} \begin{pmatrix} \xi_1(\lambda) \\ \xi_2(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \xi_1(\lambda) \\ \xi_2(\lambda) \end{pmatrix}
\]

So \( \frac{d}{d \lambda} \xi_2(\lambda) = \lambda \xi_2(\lambda) \quad \Rightarrow \quad \xi_2(\lambda) = e^{\lambda \lambda} \xi_2(\lambda_0) \)

Then \( \xi(\lambda) = \lambda \xi_1(\lambda) + \xi_2(\lambda) \)
This is an inhomogeneous first order equation.

\[
\frac{d}{dz} \left( e^{\lambda z} \xi_1(z) \right) = \xi_2(z)
\]

Method of integrating factor

\[e^{\lambda z} \xi_1(z) - \xi_1(0) = \xi_2(z)\]

Integrate from 0 to z

\[-1\int_{0}^{z} e^{\lambda x} \xi_1(x) = e^{\lambda z} \xi_1(0) + \int_{0}^{z} e^{\lambda z} \xi_2(z) d\xi_2\]

You can imagine, that for bigger blocks we will get higher order polynomials. If eigenvalues \(\lambda_1, \ldots, \lambda_m\) have multiplicities \(m_1, \ldots, m_m\)

\[U(z) = e^{z} \sum_{i=1}^{m} \frac{z^{m_i}}{m_i!} \]

where \(\psi_i(z)\) are polynomials of degree \(d_i \leq m_i - 1\), \(m_i \geq 1\).

For inhomogeneous equations

\[
\frac{d}{dz} U(z) = A U(z) + f(z)
\]

One can see that

\[U(z) = e^{Az} U(0) + \int_{0}^{z} e^{A(z-x)} f(x) dx\]
We will use the notation $L[u] = 0$, $L = \frac{d}{dt} I + P(t)$, and $Q \cdot q = \frac{d}{dt} I - A$ for the homogeneous solutions $u_1, u_2$ of the equation $L[u] = 0$ lead to linear combination $\alpha u_1 + \beta u_2$ which would also be a solution $L[u] = 0$. So the solutions form a vector space.

Given that the space of initial conditions are $n$-dimensional we expect the solution space to be $n$-dimensional as well.

Many of these observations go over to the more general homogeneous first order linear equations

$$\frac{d}{dt} u(t) + P(t) u(t) = 0$$

$P(t)$ is a matrix of size $n \times n$ with analytic entries. The previous discussion correspond to $P(t) = -A$, a constant matrix.

Now consider $n$-solutions of the linear homogeneous equation: $u_1, \ldots, u_n$.
Consider the matrix 

\[
\begin{pmatrix}
\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n
\end{pmatrix} = \mathbf{U}(z)
\]

If the \( \mathbf{u}_1, \mathbf{u}_2 \) are linearly dependent, then 

\[
W(\mathbf{u}_1(z), \ldots, \mathbf{u}_n(z), z) = \det \left[ \mathbf{u}_1(z), \ldots, \mathbf{u}_n(z) \right]
\]

would be zero for all \( z \). Thus \( W \) being non-zero somewhere is the indication that these are linearly independent solutions.

Since 

\[
\frac{d}{dz} \mathbf{u}_i(z) = \mathbf{P}(z) \mathbf{u}_i(z),
\]

consider \( \mathbf{u}_i(z+h) \) for small \( h \).

\[
\mathbf{u}_i(z+h) = \mathbf{u}_i(z) + h\mathbf{P}(z) \mathbf{u}_i(z) + O(h^2) \mathbf{u}_i(z)
\]

So 

\[
\left[ \mathbf{u}_1(z+h), \ldots, \mathbf{u}_n(z+h) \right] = \left( \mathbf{I} - h\mathbf{P}(z) + O(h^2) \right) \left[ \mathbf{u}_1(z), \ldots, \mathbf{u}_n(z) \right]
\]

Taking \( \det \)

\[
W_{z+h} = \det \left[ \mathbf{I} - h\mathbf{P}(z) + O(h^2) \right] W_z.
\]

\[
\frac{dW}{dz} = -\text{tr}(\mathbf{P}(z)) W(z)
\]
Note that
\[ det \begin{bmatrix} 1 - h P_{1,1}(2) & -h P_{1,2}(2) & \cdots & -h P_{1,n}(2) \\ -h P_{2,1}(2) & 1 - h P_{2,2}(2) & \cdots & -h P_{2,n}(2) \\ \vdots & \vdots & \ddots & \vdots \\ -h P_{n,1}(2) & -h P_{n,2}(2) & \cdots & 1 - h P_{n,n}(2) \end{bmatrix} = \left( -h P_{1,1}(2) \right) \cdots \left( 1 - h P_{n,n}(2) \right) + O(h^4) \\
= 1 - h \left( P_{1,1}(2) + \cdots + P_{n,n}(2) \right) + O(h^2) \\
= 1 - h \text{tr}(P(2)) + O(h^2) \\

Since $W$ is a scalar satisfying a first order homogeneous equation, we can solve it exactly:

\[ W(x) = e^{-\frac{1}{2h} \text{tr} P(x) + i} W(x_0) \]

If we restricted ourselves to $P(x) = -A$, then

\[ -\text{tr} P(x) = \text{tr} A = \lambda_1 + \cdots + \lambda_n \]

In the diagonalizable case, with real argument, "volumes" change like $e^{\lambda x}$.

The change of Wronskian just captures that.

\[ (U(\phi x)) = \begin{bmatrix} U_1(\phi x) & \cdots & U_n(\phi x) \end{bmatrix}, \]

In a called fundamental solution if $U(\phi x) e^{\lambda x}$ and $U(\phi x) e^{\mu x}$.

Now we take an $n$-th order differential equation!
Linear nth order differential equation is defined by the equation:

\[ L[u] = L(u^{(n)} + p_1(t)u^{(n-1)} + \cdots + p_{n-2}(t)u'' + p_{n-1}(t)u' + p_n(t)u = 0 \]

\[ L[u] = 0 \] is the homogeneous equation.

\[ L[u](t) = f(t) \] is the inhomogeneous equation.

If \[ u_1 \] is a special solution, so that

\[ L[u_1](t) = f(t) \]

and \[ u \] is a solution of the homogeneous equation:

\[ L[u+u_1](t) = f(t) \]

In fact, any solution of the inhomogeneous equation can be written this way, since \( u_1 \) is a solution to the homogeneous equation.

Now, let us write this equation as a system of first order equations with \( U = \begin{pmatrix} u & u' \\ u' & u'' \end{pmatrix} \)

\[ \frac{d}{dt} \begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \]

For simplicity, we will take \( f = 0 \), but it is not essential.

We define \( W(u_1, \ldots, u_n) = \begin{pmatrix} u_1 \\ u_1' \\ \vdots \\ u_n \\ u_n' \end{pmatrix} \)
Note that \( P(z) = \left( \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 0 & 0 \\ z & -z \end{array} \right) \). So \( \chi P(z) = P(z) \).

So \( \frac{\partial}{\partial z} W + P(z)W = 0 \)

\[
W(z) = e^{-\int P(z)dz} W(z_0)
\]

The fundamental solution \( W_{\alpha}(z_0, z) = \delta(z-z_0) \).

\[ \text{Power Series Solution.} \]

\[ U''(z) + z U(z) = 0 \quad \text{Airy equation} \]

Try \( U(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \quad \text{Call } a_0 = 1 \).

\[
2 \cdot 10 + 3 \cdot 2 a_2 z + 4 \cdot 3 a_3 z^2 + \ldots + z \left( a_3 + a_2 + a_1 + a_0 \right) = 0
\]

\[ 3 \cdot 2 a_3 + a_0 = 0, \quad 4 \cdot 3 a_4 + a_1 = 0, \quad 5 \cdot 4 a_5 + a_2 = 0, \quad 6 \cdot 5 a_6 + a_3 = 0 \]

\[ \frac{a_{n+3}}{a_n} = -\frac{1}{(n+3)(n+2)} \]

\[ U(0) = 1, \quad U'(0) = 0 \]

Note that \( U''(0) = 0 \)

\[ U(x) = 1 - \frac{1}{3} x^2 + \frac{1}{5} x^3 - \frac{1}{7} x^4 + \frac{1}{9} x^5 - \frac{1}{11} x^6 - \ldots \]

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