Solution to Problem Set 4

\[ K.E. = \frac{1}{2} m \left[ l^2 \dot{\theta}_1^2 + l^2 \dot{\theta}_2^2 \right] = \frac{m l^2}{2} \left[ \dot{\theta}_1^2 + 2 \dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2 \right] \]

\[ T = m l^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \]

Total P. E. = \[ \frac{1}{2} \ k l^2 \left[ (\dot{\theta}_1 - \dot{\theta}_2)^2 - \frac{1}{2} m g l (2 \dot{\theta}_1^2 + \dot{\theta}_2^2) \right] + \text{const} \]

\[ V = k l^2 \begin{bmatrix} 2(1-\beta) & -1 \\ -1 & 1-\beta \end{bmatrix} \]

\[ V = m \omega_e^2 l^2 \begin{bmatrix} 2(1-\beta) & -1 \\ -1 & 1-\beta \end{bmatrix} \]

a) If \( \omega \) is a resonant frequency then

\[ |V - \omega^2 T| = 0 \]

or

\[ \left| \begin{pmatrix} 2(1-\beta) & -1 \\ -1 & 1-\beta \end{pmatrix} - \frac{\omega^2}{\omega_e^2} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right| = 0 \]

Let us call \( \frac{\omega^2}{\omega_e^2} = \kappa \).
\[
\begin{vmatrix}
2(1-\beta-\tau) & -1-\tau \\
-1-\tau & 1-\beta-\tau
\end{vmatrix} = 0
\]

So
\[
2(1-\beta-\tau)^2 - (1+\tau)^2 = 0
\]
\[\Rightarrow \quad \tau^2 - 2(3-2\beta)\tau + 2(1-\beta)^2 - 1 = 0
\]
The solutions are
\[
\tau = 3-2\beta \pm \sqrt{2(2-\beta)}
\]
So
\[
\omega = \sqrt{3-2\beta \pm \sqrt{2(2-\beta)}}
\]

b) For the system to become unstable, \( \tau \) has to crossover to negative region. The threshold \( \beta \) is when the one of the \( \omega \)'s become zero.

\[
\beta = \min \left( \frac{3+2\sqrt{2}}{2+\sqrt{2}}, \frac{3-2\sqrt{2}}{2-\sqrt{2}} \right) = \min \left( \frac{2+\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2} \right)
\]
\[= \frac{2-\sqrt{2}}{2}
\]

c) Using the Raleigh function, we see that
\[
\mathbf{F} = \mathbf{F} \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]
which is proportional to the \( \mathbf{I} \) matrix, making life easy. Trying solution of the form \( \mathbf{A} e^{\mathbf{t}} \).
\[
\left| \begin{array}{c}
y + yF + y^2 I \\
y^2 \begin{pmatrix}
2(1-\beta) & -1 \\
-1 & 1-\beta
\end{pmatrix} + \frac{2}{K} Y \begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix} + m^2 Y^2 \begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}
\end{array} \right| = 0
\]

\[
\left| \begin{array}{c}
y^2 \begin{pmatrix}
2(1-\beta) & -1 \\
-1 & 1-\beta
\end{pmatrix} + \left( \frac{\xi Y}{K} + \frac{\xi^2}{\omega_0^2} \right) \begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}
\end{array} \right| = 0
\]

Notice that it is the same determinant we computed in part a), if we replace \( z \rightarrow \left( \frac{\xi Y}{K} + \frac{\xi^2}{\omega_0^2} \right) \).

That means the solutions are given by:

\[
\frac{\xi^2}{\omega_0^2} + \frac{\xi Y}{K} + 3 - 2\beta + \sqrt{\xi(2-\beta)} = 0
\]

When \( 3 - 2\beta + \sqrt{\xi(2-\beta)} \) are positive, i.e., the system is stable, and \( \xi \) is small, \( y \) is going to be complex:

\[
y = -\frac{1}{2} \pm \frac{\sqrt{\xi(2-\beta)}}{2} i
\]
The damping rate \( \frac{1}{\tau} \) in half the sum of
the two roots of the quadratic equation:
\[
\frac{1}{\tau} = \frac{F \omega_n^2}{2K},
\]
from the coefficients of
the quadratic equation. It is the same
for both the frequencies.

Since \( K = m \omega_n^2 \),
\[
\tau = \frac{F}{2m}
\]

Honestly, we don't need a calculation here.

Remember \( \gamma + \gamma^* = \frac{F}{\tau} = \frac{\gamma^* F \alpha}{\alpha^* \tau} \)
for a particular mode.

Since \( F = \frac{F}{m} \) here, \( \frac{1}{\tau} = \frac{F}{2m} \)
for all modes.