Conservation Theorems and Symmetries Principle

\[ \frac{\partial L}{\partial \dot{q}_j} = \text{could be interpreted as momenta.} \]

\[ L = \frac{1}{2} \sum_i m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - \sum_{\text{no velocities}} \]

\[ \frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i = p_i \]

\[ p_j = \frac{\partial L}{\partial \dot{q}_j} \]

canonical/conjugate momentum

Particles in an E.M. field

\[ L = \sum \frac{1}{2} m_i \dot{r}_i^2 - \sum q_i \phi (\vec{r}_i) \]

\[ + \sum q_i \vec{\nabla} \phi (\vec{r}_i) \cdot \dot{\vec{r}_i} \]

\[ \vec{p}_i = m_i \vec{\dot{r}_i} + q_i \vec{\nabla} \phi (\vec{r}_i) \]
If $L$ is independent of $q_j$

$$\dot{p}_j = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} = 0$$

$q_j$ is called cyclic/ignorable

The corresponding momentum is conserved:

$$p_j = \text{constant}$$

$$p_j(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = \text{const.}$$

In principle, one could try to transform to coordinates so that "all" coordinates become cyclic. It works out for 'integrable' systems.

Move on it later.
For conservative systems, with \( V \) fixed, coordinates only:

\[
\dot{p}_j = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} = Q_j = \sum F_i \frac{\partial r_i}{\partial q_j}
\]

Remember:

\[
\frac{\partial r_i}{\partial q_j} = \frac{\partial r_i}{\partial \dot{q}_j} = \frac{\partial v_i}{\partial \dot{q}_j}
\]

\[
\dot{q}_j = \frac{\partial T}{\partial \dot{q}_j} = \sum m_v \frac{\partial r_i}{\partial q_j}
\]

Two examples:

Translation

\[
\mathbf{F}_i \rightarrow \mathbf{F}_i + \mathbf{F}_n
\]

\[
\frac{\partial r_i}{\partial q} = \mathbf{n}
\]

\[
\mathbf{p} = (\sum m_v \mathbf{v}_i) \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{p}
\]
Rotation

\[ \text{d}r_i \cdot n \times \text{d}r_i = (\hat{n} \, d\theta) \times \text{d}r_i \]

\[ \mathbf{Q} = \sum \mathbf{F}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \theta} = \mathbf{F}_i \cdot (\hat{n} \times \text{d}r_i) \]

\[ = \hat{n} \cdot (\sum \mathbf{r}_i \times \mathbf{F}_i) = \hat{n} \cdot \mathbf{N} \]

Component of torque in \( \hat{n}'s \) direction

\[ p_\theta = \sum m_i \mathbf{v}_i \cdot (\hat{n} \times \text{d}r_i) \]

\[ = \hat{n} \cdot \sum m_i \mathbf{r}_i \times \mathbf{v}_i \]

\[ = \hat{n} \cdot \mathbf{L} \]

Conservation laws result of symmetry

\[ L = \frac{1}{2} \sum m_i \mathbf{v}_i^2 + \sum V_i(r_i) \quad \Rightarrow \quad \mathbf{p}_\theta \text{ conserved} \]

\[ = \frac{1}{2} \sum m_i \mathbf{v}_i^2 + \sum V_i(r_i) + \sum V_i(r_i) \quad \Rightarrow \quad \mathbf{L} \text{ conserved} \]
Energy conservation and time translation invariance

\[ \frac{dL}{dt} = \sum_j \left( \frac{\partial L}{\partial q_j} \dot{q}_j + \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j \right) + \frac{eL}{\kappa} \]

\[ = \sum_j \left( \frac{1}{d+1} \left( \frac{\partial T}{\partial q_j} \dot{q}_j + \frac{\partial T}{\partial \dot{q}_j} \ddot{q}_j \right) \right) + \frac{eL}{\kappa} \]

\[ = \frac{d}{dt} \sum_j \frac{eL}{\kappa} \frac{\partial}{\partial \dot{q}_j} \dot{q}_j + \frac{eL}{\kappa} \]

\[ \frac{d}{dt} \mathbf{h} = -\frac{\partial L}{\partial \dot{q}_j} \]

\[ \mathbf{h} = \sum_j \frac{eL}{\kappa} \frac{\partial}{\partial \dot{q}_j} \dot{q}_j - \mathbf{L} \quad \text{is the Hamiltonian / Energy function} \]

If \( L \) has no explicit time dependence

\[ \mathbf{h} = \text{constant} \quad (\text{Jacobi's integral}) \]
Remember Rayleigh dissipation function?

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = -\frac{\partial F}{\partial q_j} \]

Some derivation

\[ \frac{dh}{dt} = -\frac{\partial L}{\partial t} - \sum \frac{\partial F}{\partial q_j} \]

If $F$ is a homogeneous function of degree 2 in $q$'s

[Example: $\sum f(q_i q_j)$, but also $\sum \frac{\partial F}{\partial q_i} q_i q_j$]

Then

\[ \sum \frac{\partial F}{\partial q_j} q_j = 2F \]

So

\[ \frac{dh}{dt} = -2F - \frac{\partial L}{\partial t} \]

When $\frac{\partial L}{\partial t} = 0$
\[ \frac{dh}{dt} = -2F \]

Change in energy

\[ = -2 \int F \, dt \]

Interesting example: circuits

q - charge \quad i = \dot{q}

\[ L = L_0 - \frac{q^2}{2C} \]

\text{Contr. from Inductance (like K.E.)}

\text{Contr. from Capacitance (like P.E.)}

\[ F = \frac{1}{2} R \dot{q}^2 \]