Hamilton–Jacobi Theory

Time evolution is a canonical transformation:

\[ q = q(q_0, p, t) \]
\[ p = p(q_0, p_0, t) \]

Final positions/momenta in terms of initial ones.

If we could go from \((q, p) \rightarrow (q=q_0, p=p_0)\) or some func of \(q_0, p_0\)
then \(q=0, p=0\)

\[ \dot{q} = \frac{\partial K}{\partial p}, \quad \dot{p} = -\frac{\partial K}{\partial q} \]

Having \(K=0\) achieves that!

If there is a generating function \(S\) of the second kind
\[ K = H(q, p, t) + \frac{\partial S}{\partial t} = 0 \]
\[ p_i = \frac{\partial S}{\partial q_i} \]
So we have

\[ H(q,\frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0 \]

Hamilton–Jacobi equation!

If \( H \) is independent of time

\[ H(q,\frac{\partial S}{\partial q}) = \alpha = -\frac{\partial S}{\partial t} \]

So, try

\[ S(q, p, t) = W(q, p) = \alpha t \]

\[ H(q, \frac{\partial W}{\partial q}) = \alpha \]

\( S \rightarrow \) Hamilton's principal function

\( W \rightarrow \) Hamilton's characteristic function
In general \[ S(q_1, \ldots, q_n; \alpha, \ldots, \omega; t) \]

\[ \frac{\partial S(g, p, t)}{\partial p_i} = \frac{\partial S(g, \alpha, t)}{\partial q_i} \]

\[ q_i = p_i = \frac{\partial S(g, p, t)}{\partial p_i} = \frac{\partial S(g, \alpha, t)}{\partial q_i} \]

\[ p_i = \frac{\partial S(g, \alpha, t)}{\partial q_i} \]

Use these to solve \( g(q, \beta, t) \) \( p(q, \beta, t) \).

For time-independent case: \( q_1, q_2, \ldots, q_n \) are related. Any \( n \) of them could be used as indep.:

D) Easy example: Free particle

\[ H = \frac{p^2}{2m}, \quad S = W(x, y, z; p_x, p_y, p_z) - \beta t \]

\[ \sum \frac{(\partial_x W)^2 + (\partial_y W)^2 + (\partial_z W)^2}{2m} = \beta \]

\[ W = p \cdot x \quad \text{with} \quad \frac{p^2}{2m} = \beta \]

Of course \( p = \tilde{p} \).
\[ S = \mathbf{p} \cdot \mathbf{x} - \frac{p^2}{2m} t \]

\[ Q_x = \frac{\partial S}{\partial p_x} = x - \frac{p_x}{m} + t \]

\[ Q_x = x(t) - \omega^2 t = x(0) \]

2) Next easy example: Harmonic Oscillator

Use characteristic function \( W(\theta) \)

\[ H(\theta) = \frac{p^2}{2m} + \frac{m \omega^2 q^2}{2} = E \]

\[ \frac{1}{2m} \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{m \omega^2 q^2}{2} + \frac{2S}{2} = 0 \]

or

\[ \frac{1}{2m} \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{m \omega^2 q^2}{2} = \alpha \]

\[ W = \int dq \sqrt{2m \omega^2 - p^2} \]

and

\[ S = \int dq \sqrt{2m \omega^2 - p^2} - \alpha t \]
Now, we could either use $P$ or use $\alpha$.

Let's take $P = \alpha$

\[
Q = \beta = \frac{\partial S}{\partial P} = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{mdQ}{2ma - m^2a^2q^2}} - t
\]

\[
= \frac{1}{w} \sin^{-1} \left( \frac{m^2a^2}{2a} \right) - t
\]

\[
Q = \frac{2x}{m \omega^2} \sin \left( \omega t + \beta \right)
\]

Solution of harmonic osc. once more!

In general, solving such first order equation required finding characteristic curves, which in turn is related to finding solutions of Hamilton's $\delta S/\delta q = 0$. So we back to square one!

Before we go on:

\[
S = \frac{\partial S}{\partial \dot{q}} \dot{q} + \frac{\partial S}{\partial \dot{p}} \dot{p} + \frac{\partial S}{\partial t} = 0. \quad H = L
\]

For $\frac{\partial S}{\partial \dot{q}} = 0$:

\[
W = F \dot{q}
\]

So $S = \int dt$ on ballo $W = F \dot{q}$ along ball.
Action-Angle Variables

Consider a one variable periodic motion with $H = H(q, p)$

Consider the variable $J(t) = \int p dq$

$J$ is obviously a constant. The orbits are constant energy curves. So $H = H(J)$

Use a characteristic function

$F_2 = W = W(q, J)$, $\frac{\partial W}{\partial q} = p$

The conjugate variable to $J$,

$\omega = \frac{\partial W}{\partial J}$

Since $H = H(J)$

$\omega = \frac{\partial H}{\partial J} = v(J)$

constant on the trajectory.

Total change of $\omega$ through the period

$\Delta \omega = \int_{\phi} \frac{\partial \omega}{\partial q} dq = \int_{\phi} \frac{\partial W(q, J)}{\partial q} dq = \int_{\phi} \frac{\partial W(q, J)}{\partial q} dq$
\[ \frac{2}{25} \delta_1 \delta_2 = \frac{2 \sqrt{5}}{25} = 1 \]

So \( \Delta \omega = 1 = \frac{2 \pi}{2} \text{ time period} \)

\( \theta(2\pi \theta) \) is like an angle variable

One more, back to harmonic osc.

\[ J = \oint \phi \, dq = \oint \sqrt{2ma - m\omega^2} \, dq \]

Substitute \( q = \sqrt{\frac{2a}{m\omega^2}} \sin \theta \)

\[ J = \sqrt{\frac{2a}{m\omega^2}} \oint \phi \cos \theta \, d(\sin \theta) \]

\[ = \frac{2\pi a}{\omega} \int_0^{\pi} \cos^2 \theta \, d\theta \]

\[ = \frac{2\pi a}{\omega} \]

\[ \alpha = \frac{J}{2\pi} \]

\[ \dot{\nu} = \frac{\delta H}{\nu} = \frac{\omega}{2\pi} \]
\[ \omega = \frac{\omega + \beta}{\sqrt{\omega}} \]

We already saw \[ W = \int dq \sqrt{2m \cdot m \cdot \omega^2 q^2} \]

\[ = \int dq \frac{m \cdot \omega}{\sqrt{\frac{m \cdot \omega}{2} - \omega^2 q^2}} \]

\[ \omega = \frac{m \omega}{2i} \int dq \frac{1}{\sqrt{m \omega (\frac{i}{\hbar} - \omega q^2)}} = \frac{1}{2\pi} \int dq \frac{1}{\sqrt{\frac{i}{\hbar} \omega - q^2}} \]

\[ = \frac{1}{2\pi} \sin^{-1} \left( \frac{q}{\sqrt{\frac{i}{\hbar} \omega}} \right) \]

\[ q = \sqrt{\frac{1}{\pi \hbar} \sin 2\pi \omega} \]

For a completely separable system

\[ W(q, a_1, \ldots, a_n) = \sum_i W(q_i, a_1, \ldots, a_n) \]

Action \[ J_i = \oint p_i \, dq_i \quad (\text{no summation over } \var) \]

\[ = \oint \frac{\partial W}{\partial q_i} \, dq_i \quad \text{(convention)} \]

Angular \[ \omega^i = \frac{\partial W}{\partial J_i} = \sum_i \frac{\partial W}{\partial q_i} (q, a_1, \ldots, a_n) \]
\[
\varphi_i = \frac{\partial \mathcal{H}(\varphi_1, \ldots, \varphi_n)}{\partial \varphi_i} = \Omega_i.
\]

An interesting example is motion due to central force in a plane.

\[
H = \frac{1}{2m} \left( \frac{p_r^2}{r^2} + \frac{p_\theta^2}{r^2} \right) + V(r)
\]

\(l = p_\theta\) is a constant of motion

\[
\int p_\theta \, d\theta = 2\pi p_\theta = J_\theta
\]

\[
H = \alpha + \sqrt{2m \left( \alpha - V(r) - \frac{J_\theta^2}{4m^2 r^2} \right)}
\]

\[
J_r = \int \sqrt{2m \left( \alpha - V(r) - \frac{J_\theta^2}{4m^2 r^2} \right)} \, dr
\]

In principle, this relation could be used to express \(\alpha = \mathcal{H}^{-1}(J_\theta, J_r)\)

\[
W = \int p_\theta \, d\theta + \int p_r \, dr
\]

\[
= \frac{1}{2\pi} \int \left( \arctan(x) + \sqrt{2m \left( x - V(r) - \frac{J_\theta^2}{4m^2 r^2} \right)} \right) \, dr
\]

\[
= \frac{1}{2\pi} (\theta + \sqrt{2m \left( x - V(r) - \frac{J_\theta^2}{4m^2 r^2} \right)}) \, dr
\]
\[ w_r \propto \int \frac{dr}{\sqrt{2m(U - V(r) - \frac{J_0^2}{4r^2})}} \propto t \]

\[ \omega_0 = \Omega - \text{additional term that are } r \text{ dependent...} \]

\[ \uparrow \]

\[ \text{does not change uniformly in time} \]

\[ \text{except for circular orbits} \]

\[ \Rightarrow \]

\[ \text{Compensation from } r \text{ dependent terms.} \]

Usefulness in perturbation theory

\[ \Phi_{\nu} = \sum \alpha_{j_1 \ldots j_n} e_{\nu_{j_1 \ldots j_n}} \]

\[ \text{Usefulness in perturbation theory} \]
Adiabatic Invariants

Pendulum with varying length

How does its angle amplitude evolve in time?

\[ H(\theta, p, t) \]

\[ \mathcal{L} = \dot{\theta} \left( p \dot{\theta} + \Gamma \right) \]

\[ \mathcal{L} = \dot{p} \left( \theta \right) \]

\[ \int_0^T \dot{p} d\theta = 0 \]

\[ \int \dot{p} d\theta = 0 \]

\[ \text{Euler-Lagrange equation of motion} \]

\[ \text{Strait between trajectories} \]

\[ \Rightarrow (\oint p \, dq_i - \int_0^\infty p \, dq) = 0 \]

\[ \Rightarrow (\oint p \, dq_i - \int_0^\infty p \, dq) = 0 \]
$J_i \approx J_f$
That implies $E$ is fixed, for harmonic oscillators.

Used in Bohr-Sommerfeld quantization

$$J = n\hbar$$

This is because wave functions are like

$$\psi(x) \propto e^{i\frac{Jx}{\hbar}}$$

Semiclassical

$W \rightarrow W + 1$ is the same point in phase space.

$$\Rightarrow J = n\hbar$$