1. Goldstein Ch. 4, problem 6.
Let $A$ and $B$ be rotations in parts a) and b), respectively and let $R_z$ be a regular rotation about the current $z$-axis. The rotation about the old $z$ axis is

$$C = (BA)R_z(BA)^{-1}$$

because we first go back with $(BA)^{-1}$ so that the $z$-axis is the current $z$-axis, do a regular rotation around it and then restore with $BA$.

Thus

$$CBA = (BA)R_z(BA)^{-1}BA = BAR_z,$$

which is the sequence of rotations that defines Euler’s angles.

2. Goldstein Ch. 4, problem 7.
Consider e.g. a rotation through $\pi$ about the $z$-axis. $A$ is a diagonal matrix with entries $1, -1, -1$ equivalent to reversing the $x$ and $y$ axes. We get

$$P_+ = \text{DiagonalMatrix}(1, 0, 0), \quad P_- = \text{DiagonalMatrix}(0, 1, 1)$$

Squaring a diagonal matrix means squaring the diagonal matrix elements. We see that $P^2 = P \pm$.

$P_\pm$ are projection operators. $P_+$ projects out the $z$ component of any vector $\mathbf{F}$, while $P_-$ takes its component in the $xy$ plane (the part perpendicular to the $z$-axis).

Let $z$ be the rotation axis and let the two planes make angles $\alpha$ and $\beta$ with the $x$-axis. It is sufficient to look at reflections in the $xy$ plane. Under the first reflection the $x$ axis rotates by $2\alpha$ and reverses its direction. It now makes an angle $\beta - 2\alpha$ with the second plane. Reflecting through the second plane restores the direction and additionally rotates by $2(\beta - 2\alpha)$. The net result is a rotation by

$$\Phi = 2\alpha + 2(\beta - 2\alpha) = 2(\beta - \alpha).$$

The same works for the $y$-axis. In this case $\alpha \rightarrow \alpha - \pi/2$, $\beta \rightarrow \beta - \pi/2$, but the net result is still a rotation by the same $\Phi$. We see that the angle between planes must be $\beta - \alpha = \Phi/2$ to obtain a rotation by $\Phi$.

4. Goldstein Ch. 4, problem 10.

a) \[ e^B e^C = \left( \sum_{n=0}^{\infty} \frac{B^n}{n!} \right) \left( \sum_{m=0}^{\infty} \frac{C^m}{m!} \right) = \sum_{N=0}^{\infty} \sum_{m+n=N} \frac{N!}{n!m!} B^n C^m = \sum_{N=0}^{\infty} \frac{(B+C)^N}{N!} = e^{B+C} \]

The commutation is used in writing

$$ (B + C)^N = \sum_{m+n=N} \frac{N!}{n!m!} B^n C^m,$$
where all $B$'s have been commuted to the left of $C$'s.

b) This follows from a) by setting $C = -B$.

c) First note that

$$(C B C^{-1})^n = C B C^{-1} C B C^{-1} C B C^{-1} \ldots C B C^{-1} = C B^n C^{-1}.$$ 

We have

$$\epsilon^{C B C^{-1}} = \sum_{n=0}^{\infty} \frac{(C B C^{-1})^n}{n!} = \sum_{n=0}^{\infty} \frac{C B^n C^{-1}}{n!} = C \left( \sum_{n=0}^{\infty} \frac{B^n}{n!} \right) C^{-1} = C A C^{-1}$$

where $A$ is orthogonal.

5. Goldstein Ch. 4, problem 14.

a) $i$ and $j$ must be distinct, same for $r$ and $m$. Further among $i, j, r, m$ there can be only 2 distinct indices, otherwise (if there are 3), $p$ will have to be equal to one of them and each term in the summation will be zero. Thus we must have either $r = i, m = j$ or $r = j, m = i$. In the first case $\epsilon_{ijp}$ and $\epsilon_{rmp}$ have the same sign and their product is 1, in the second case – opposite sign with product equal to $-1$. Thus

$$\epsilon_{ijp} \epsilon_{rmp} = \delta_{ir} \delta_{jm} - \delta_{im} \delta_{jr}.$$ 

b) Performing a cyclic permutation of indices in the above equation, we obtain

$$\epsilon_{pij} \epsilon_{prm} = \delta_{ir} \delta_{jm} - \delta_{im} \delta_{ji}$$

Now contracting the second index, i.e. setting $r = i$, we get (note that $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$)

$$\epsilon_{pij} \epsilon_{pim} = 3\delta_{jm} - \delta_{im} \delta_{ji} = 3\delta_{jm} - \delta_{jm} = 2\delta_{jm}$$