1. Goldstein Ch. 3, problem 18.

Impulse $S$ means an instantaneous change in particles’ linear momentum

$$\Delta p = S \hat{r}$$

The corresponding changes in the energy, angular momentum and Runge-Lentz vector are

$$\Delta E = \frac{S^2}{2m}, \quad \Delta L = 0, \quad \Delta A = \Delta p \times L = -Sl\hat{\theta},$$

where $-\hat{\theta}$ is in the clockwise direction.

I will use the following 3 formulas from Goldstein

$$l^2 = mka(1 - e^2) \quad (1)$$

$$a = -\frac{k}{2E} \quad (2)$$

$$A = mke \quad (3)$$

Note that the change in the Runge-Lentz vector is perpendicular to its initial direction along the semimajor axis. The magnitude of the Runge-Lentz vector for the new orbit thus is

$$A'^2 = A^2 + S^2l^2$$

and from Eq. (3) we get

$$e'^2 = e^2 + \frac{S^2l^2}{(mk)^2} = e^2 + \frac{S^2}{mk}a(1 - e^2),$$

where I used Eq. (1). Note that $S^2/(2m)$ has dimensions of inverse length. Define

$$a_0 \equiv \frac{mk}{S^2}.$$

The above equation for the new eccentricity $e'$ then reads

$$e'^2 = e^2 + \frac{a}{a_0}(1 - e^2)$$

The semimajor axis rotates clockwise by an angle $\alpha$ such that

$$\tan \alpha = \frac{|\Delta A|}{A} = \frac{Sl}{A} = \frac{Sl}{mke} = \sqrt{\frac{a(1 - e^2)}{a_0e^2}},$$

where I used Eq. (1) to express $l^2$ in terms of known quantities.

Finally, Eq. (2) implies

$$\frac{S^2}{2m} = E' - E = \frac{k}{2} \left( \frac{1}{a} - \frac{1}{a'} \right)$$
resulting in
\[
\frac{1}{a'} = \frac{1}{a} - \frac{1}{a_0}, \quad a' = \frac{aa_0}{a_0 - a}
\]
Note that we must have \( a_0 > a \) for the orbit to remain elliptical after the impulse.

2. Goldstein Ch. 3, problem 29.

\[
A = p \times L - mk\dot{r}
\]
\[
L \times A = pl^2 - mkL \times \dot{r}
\]
Let \( A = A\hat{x} \) and \( L = l\hat{z} \). The above equation now reads
\[
lA\hat{y} = pl^2 - mk\ell\dot{\theta}
\]
\[
p - \frac{A}{l}\hat{y} = \frac{mk}{\ell}\dot{\theta}
\]
\[
\left(p - \frac{A}{l}\hat{y}\right)^2 = \left(\frac{mk}{\ell}\right)^2 = R^2 = \text{const},
\]
proving the statements of the problem.

The potential is \( V = \frac{k}{2r^2} \). Let’s plug this into Eqs. (3.94) and (3.95) from Goldstein
\[
\Psi = \int_{r_m}^\infty \frac{dr}{r^2 \sqrt{\frac{2mE}{r^2} - \frac{mk}{l^2} - \frac{1}{r^2}}}
\]
where \( u = 1/r \),
\[
a^2 = \frac{2mE}{l^2}, \quad b^2 = 1 + \frac{mk}{l^2} = 1 + \frac{k}{2Es^2}
\]
and I used \( l = smv_0 = s\sqrt{2mE} \).

Let \( u = a/b\sin\alpha \). \( u = 0 \) corresponds to \( \alpha = 0 \), \( u = u_m \) (closest approach) – to \( \alpha = \pi/2 \). Indeed, the denominator in the expression for \( \Psi \) comes from \( \dot{r} \) because this equation obtains from dividing \( \dot{\theta} \) by \( \dot{r} \) and then integrating. At closest approach \( \dot{r} \) and therefore the denominator vanish meaning \( \alpha = \pi/2 \). Thus
\[
\Psi = \int_0^{\pi/2} \frac{a/b\cos\alpha d\alpha}{a \cos \alpha} = \frac{\pi}{2b}
\]
\[
\Theta = \pi - 2\Psi = \pi \left(1 - \frac{1}{b}\right), \quad \sin \Theta = \sin \pi x
\]
\[
x = \frac{\Theta}{\pi} = 1 - \frac{1}{b}
\]
From this and the expression for \( b^2 \) above, we obtain the impact parameter
\[
s^2 = \frac{k}{2Eb^2} \frac{1}{1} = \frac{k (1 - x)^2}{2E (2x - x^2)}
\]
The differential cross section obtains by taking the differential of the last expression and plugging into Eq. (3.93) of Goldstein
\[
\sigma(\Theta)d\Theta = \frac{1}{\sin \Theta} |sds| = \frac{k}{2E} \frac{1}{\sin \pi x x^2(2-x)^2} (1-x)dx
\]

4. Goldstein Ch. 3, problem 35.
It’s convenient to use the conservation (for $r < a$) of the Runge-Lentz vector $\mathbf{A}$ in this problem. This vector is along the line connecting the center of force with the point of closest approach. Let the $x$-axis be along the incident momentum and consider a particle moving in the $xy$ plane. Then the angle $\Psi$ between the above line and the incident direction can be found from
\[
\tan \Psi = \frac{|A_y|}{|A_x|}
\]
while the scattering angle is $\Theta = \pi - 2\Psi$.

To determine the components of the Runge-Lentz vector, let’s evaluate it at the point of entry into the $1/r$ potential, i.e. at $r = a$. We have
\[
\mathbf{p} = mv_0 \hat{x} = p_0 \hat{x}, \quad \mathbf{L} = -p_0 s \hat{z},
\]
\[
\mathbf{A} = \mathbf{p} \times \mathbf{L} + mk \hat{r} = p_0^2 s \hat{y} + km (-\hat{x} \cos \alpha + \hat{y} \sin \alpha) = -\hat{x} km \cos \alpha + \hat{y} (km \sin \alpha + p_0^2 s),
\]
where $\alpha$ is the angle $\hat{r}$ makes with the negative $x$-axis. Using
\[
\sin \alpha = \frac{s}{a}, \quad \cos \alpha = \frac{\sqrt{s^2 - a^2}}{a}, \quad \frac{p_0^2 \sin \alpha}{km} = \frac{2E}{k}
\]
we obtain
\[
\tan \Psi = \frac{km \sin \alpha + p_0^2 s}{km \cos \alpha} = \frac{s}{\sqrt{a^2 - s^2}} \left(1 + \frac{2Ea}{k}\right), \quad \Theta = \pi - 2 \tan^{-1} \left[\frac{s}{\sqrt{a^2 - s^2}} \left(1 + \frac{2Ea}{k}\right)\right]
\]
Solving this equation for $s^2$ yields
\[
s^2 = \frac{a^2(1+x)}{1 + \eta + (1-\eta)x}, \quad x \equiv \cos \Theta, \quad \eta \equiv \left(1 + \frac{2Ea}{k}\right)^2.
\]
In deriving this I used $\Theta = \pi - 2\Psi$ and
\[
\tan^2 \Psi = \frac{1 - \cos 2\Psi}{1 + \cos 2\Psi} = \frac{1 + \cos \Theta}{1 - \cos \Theta}.
\]

The differential cross section is
\[
\sigma(\Theta) = \frac{s}{\sin \Theta} \left| \frac{ds}{d\Theta} \right| = \frac{1}{2} \left| \frac{ds^2}{dx} \right| = \frac{\eta a^2}{(1 + \eta + (1-\eta)x)^2}.
\]
As a check of our calculation let’s compute the total cross section from this
\[
\sigma_T = 2\pi \int_0^\pi \sigma(\Theta) \sin \Theta d\Theta = 2\pi \int_{-1}^1 \sigma(x) dx = \pi a^2.
\]
This result is expected – the potential scatters all particles with impact parameters $s \leq a$, therefore the total cross section by definition is
\[
\sigma_T = \frac{I \pi a^2}{I} = \pi a^2.
\]