1. Goldstein Ch. 3, problem 13.

a) Let $f(r)$ be the force and $V(r)$ the corresponding potential. The component of the force towards the center provides the centripetal acceleration

$$ f(r) = \frac{mv^2}{R} = \frac{2[V(r) - E]}{R}, $$

where $R$ is the radius of the circle, $E$ is the total energy and we used the energy conservation $mv^2/2 + V(r) = E$.

$$ -\frac{dV}{dr} = f = \frac{4V}{r} - \frac{4E}{r} $$

The general solution of this ODE is

$$ V = -\frac{k}{r^4} + E \quad (1) $$

The force being the derivative of $V$ varies as the inverse fifth power of $r$.

b) The potential $V$ and the energy $E$ are in principle defined up to an arbitrary additive constant. Our convention however is to set this constant so that $V \to 0$ as $r \to \infty$. Then Eq. (1) implies $E = 0$.

c) Energy conservation implies

$$ \frac{mv^2}{2} + V = E = 0, \quad v = \sqrt{-\frac{2V}{m}} = \sqrt{\frac{2k}{mr^4}} \quad (2) $$

The angular velocity is $d\theta/dt = v/R$. Therefore

$$ dt = \frac{R}{v} d\theta = R \sqrt{\frac{m}{2k}} r^2 d\theta $$

The distance to the force center is $r = 2R \sin(\theta/2)$.

$$ dt = 4R^3 \sqrt{\frac{m}{2k}} \sin^2 \frac{\theta}{2} d\theta = R^3 \sqrt{\frac{2m}{k}} (1 - \cos \theta) d\theta. $$

The period is

$$ T = R^3 \sqrt{\frac{2m}{k}} \int_0^{2\pi} (1 - \cos \theta) d\theta = 2\pi R^3 \sqrt{\frac{2m}{k}} $$

b) Let the center of the circle be the origin of the coordinate system and the center of force be on the $x$-axis at $x = R$. Eq. (2) implies

$$ v \propto \frac{1}{r^2} = \frac{1}{4R^4 \sin^2 \frac{\theta}{2}}, \quad  \dot{x} = -v \sin \theta \propto \frac{\cos \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}}, \quad  \dot{y} = v \cos \theta \propto \frac{\cos \theta}{\sin^2 \theta}. $$

All three quantities diverge as $\theta \to 0$, i.e. as the particle goes through the center of force.
2. Goldstein Ch. 3, problem 19.

a) The effective potential is

\[ V_{\text{eff}}(r) = -\frac{k}{r}e^{-r/a} + \frac{l^2}{2mr^2} \]

Let’s make things dimensionless for convenience – let’s measure the potential in units of \( k/a \) and the distance in units of \( a \)

\[ \frac{V_{\text{eff}}(r)}{k/a} \equiv v_{\text{eff}}(x) = -\frac{e^{-x}}{x} + \frac{\alpha}{x^2}, \quad x = r/a, \]

where

\[ \alpha = \frac{l^2}{2mka}. \]

For \( \alpha \ll 1 \) the first term \( (e^{-x}/x) \) dominates at intermediate values of \( x \). The potential goes to \( +\infty \) as \( 1/x^2 \) at very small \( x < \alpha \), then becomes negative at intermediate values of \( x \). The second term takes over at larger \( x \) when \( e^{-x} \approx \alpha \), the potential increases, becomes positive and finally approaches 0 from above as \( 1/x^2 \) at large \( x \). We see that in this case the potential should have a minimum and a maximum and \( x_{\text{max}} > x_{\text{min}} \). For \( V_{\text{min}} \leq E \leq V_{\text{max}} \) the motion is bounded with two turning points; for \( E > V_{\text{max}} \) the motion is unbounded and the particle escapes to infinity from the Yukawa potential.

For \( \alpha \gg 1 \) the second term dominates at all \( x \) and the potential has no extrema. The motion is always unbounded.

This analysis shows that there must be a critical value \( \alpha_0 \) such that for \( \alpha < \alpha_0 \) the potential has a minimum and a maximum and bounded orbits are present, while for \( \alpha > \alpha_0 \) there are no extrema and all orbits are unbounded.

Differentiating \( v_{\text{eff}}(x) \) with respect to \( x \) yields the condition for extrema

\[ (x^2 + x)e^{-x} = 2\alpha \]  \( (3) \)

The left hand side of this equation is shown in Fig. 1. It reaches a maximum at

\[ x = x_0 = \frac{\sqrt{5} + 1}{2} \approx 1.62, \]

The value at the maximum is the critical value of \( \alpha \)

\[ \alpha_0 = \frac{(x^2 + x)e^{-x}}{2} \approx 0.42 \]

For \( \alpha < \alpha_0 \) the line \( y = 2\alpha \) intersects the left hand side of Eq. (3) twice. The smaller value, \( x = x_1 < x_0 \) corresponds to a minimum, the larger to a maximum. For \( \alpha > \alpha_0 \) there are no extrema. For \( \alpha = \alpha_0 \) there is a single saddle point.

Plots of the potential for \( \alpha < \alpha_0 \), \( \alpha = \alpha_0 \) and \( \alpha > \alpha_0 \) are displayed in Figs. 2, 3, and 4, respectively.

b) There seems to be a typo in Goldstein in this part – the answer \( \pi \rho/a \) seems incorrect.

According to Eqs. (3.45) and (3.46) of Goldstein for small deviations from a circular orbit

\[ u = u_0 + a \cos \beta \theta, \quad \beta^2 = 3 + \frac{rf'}{f} \]
I verified these equations and they are correct. The closest approaches to the center of force occur when

\[ \beta \theta = 2\pi n \]

In our case we get

\[ f = \frac{d}{dx} \left( \frac{e^{-x}}{x} \right) = -\left( \frac{1}{x^2} + \frac{1}{x} \right) e^{-x}, \quad f' = \left( \frac{2}{x^3} + \frac{2}{x^2} + \frac{1}{x} \right) e^{-x} \]

\[ \beta^2 = 3 + \frac{xf'}{f} = 1 - \frac{x^2}{x + 1} \]

Here \( x = x_1 < x_0 \) is the dimensionless radius of the circular orbit, i.e. \( x = \rho/a \). This circular orbit corresponds to the minimum of the potential and is stable. The one that corresponds to the maximum is unstable.

As \( x \) varies from 0 to \( x_0 \), \( \beta \) changes from 1 to 0. It is meaningful to talk about an advance of the apsides per revolution only when \( \beta \) is sufficiently close to 1. Perigees (closest approaches to the center) occur in intervals \( \Delta \theta = 2\pi/\beta \) and when, for example, \( \beta \) is very small the next one occurs only after approximately \( 1/\beta \gg 1 \) revolutions. In this case the notion of an advance per revolution doesn’t make much sense.

\( \beta \) is close to 1 for small \( x \). Then, \( \beta \approx 1 - x^2/2 \). After one revolution apsides advance by

\[ \frac{2\pi}{\beta} - 2\pi \approx \pi x^2 = \pi(\rho/a)^2 \]
Equations of motion are

$$\frac{m \dot{r}^2}{2} + \frac{l^2 + 2mh}{2mr^2} - \frac{k}{r} = E, \quad mr^2 \dot{\theta} = l$$

In a frame rotating in the same direction with angular speed $\omega$, $\dot{\theta} \rightarrow \dot{\theta} - \omega$, $r \rightarrow r$. The second equation of motion becomes

$$mr^2 \dot{\theta} = l + mr^2 \omega \equiv l'$$

If we now choose $\omega$ so that

$$l'^2 \equiv (l + mr^2 \omega)^2 = (l^2 + 2mh),$$

i.e.

$$mr^2 \omega = \sqrt{l^2 + 2mh} - l$$

equations of motion in the rotating frame are equivalent to the Kepler problem with the same energy $E$ and angular momentum $l'$. Note that $\omega$ in general is time dependent because it varies as $1/r^2$ and $r$ changes in time.

For small $h$ expanding the last equation in $h$ and using $r^2 = l/(m\dot{\theta})$, we obtain

$$\omega = \frac{h}{lr^2} = \frac{h m \dot{\theta}}{l}$$

Integrating over the period yields the average frequency – the frequency of the precession of the orbit as a whole

$$\bar{\omega} = \frac{hm \cdot 2\pi}{l^2 \cdot \tau} = \frac{h}{ka} \cdot \frac{kma \cdot 2\pi}{l^2 \cdot \tau} = \eta \cdot \frac{kma \cdot 2\pi}{l^2 \cdot \tau}$$
The quantity \( kma/l^2 \) obtains from the equation of the orbit \( 1/r = mk/l^2(1 + e \cos \theta) \). Setting \( r = a, \theta = 0 \) (closest approach), we get

\[
\frac{mka}{l^2} = \frac{1}{1 + e},
\]

\[
\eta = \frac{\bar{\omega} \tau}{2\pi} (1 + e).
\]

Plugging the numbers for Mercury yields

\[
\eta \approx 9 \times 10^{-8}.
\]

4. Goldstein Ch. 3, problem 22.
Because a change in \( l \) changes also the equation for \( \dot{\theta} \) in accord with the change in the equation for \( \dot{r} \), while the \( h/r^2 \) term in the potential doesn’t. The precession compensates for this missing change in \( \dot{\theta} \).

5. Goldstein Ch. 3, problem 23.
Use Eq. (3.74) of Goldstein

\[
\tau \approx \frac{2\pi a^{3/2}}{\sqrt{Gm}}.
\]

The ratio of masses is

\[
\frac{m_S}{m_E} = \frac{a_S^3 \tau_E^2}{a_E^3 \tau_S^2} \approx 3 \times 10^5
\]