1. Goldstein Ch. 1, problem 12.

\[ \frac{mv^2}{2} - G \frac{Mm}{R} = 0 \]

escape condition

\[ v = \sqrt{2GM/R} = \sqrt{2gR} \approx \sqrt{9.8 \times 6.4 \times 10^6} \text{ m/s} = 11.2 \text{ km/s}, \]

where we used \( mg = GmM/R^2 \), which implies \( GM/R = gR \).

2. Goldstein Ch. 1, problem 14.

Let the circle be in the \( xy \) plane with its center at the origin. Convenient coordinates are e.g. the angles that the radius to the center of the rod and the rod itself make with the \( x \)-axis. Let these angles be \( \phi \) and \( \alpha \), respectively. Coordinates of the two point masses are

\[ x_{1,2} = a \cos \phi \pm \frac{l}{2} \cos \alpha, \quad y_{1,2} = a \sin \phi \pm \frac{l}{2} \sin \alpha \]

The kinetic energy is

\[ T = \frac{m}{2} \sum_{i=1}^{2} (\dot{x}_i^2 + \dot{y}_i^2) = ma^2 \dot{\phi}^2 + \frac{ml^2}{4} \ddot{\alpha}^2 \]


Convenient coordinates are e.g.: \( x \) – the length of the segment of the string on the table and \( \phi \) – the angle that segment makes with any fixed line in the plane of the table. The hanging part of the string is then \( l - x \). The velocity of mass \( m_1 \) has two perpendicular components: \( \dot{x} \) along the radius and \( v_n = x \dot{\phi} \) normal to the radius. The velocity of mass \( m_2 \) is \( -\dot{x} \). The Lagrangian is

\[ L = \frac{m_1 + m_2}{2} \dot{x}^2 + \frac{m_1 x^2 \dot{\phi}^2}{2} + m_2 g (l - x) \]

The Lagrange equations are

\[ (m_1 + m_2) \ddot{x} = m_1 x \dot{\phi}^2 - m_2 g \]

\[ \frac{d}{dt} \left( m_1 x^2 \dot{\phi} \right) = 0 \]

The first equation says that the gravity \( (m_2 g) \) accelerates both masses along the string \( (-m_1 \ddot{x} - m_2 \ddot{x}) \) and also provides centripetal acceleration for \( m_1 \) \( (m_1 v_n^2/x = m_1 x \dot{\phi}^2) \).

Eq. (2) implies

\[ m_1 x^2 \dot{\phi} = M_z = \text{const}, \quad \dot{\phi} = \frac{M_z}{m_1 x^2} \]

This is conservation of the \( z \)-component of the total angular momentum. It’s conserved because the problem is symmetric with respect to rotations around the \( z \)-axis.
Plugging $\dot{\phi}$ into Eq. (1), we get
\[(m_1 + m_2)\ddot{x} = \frac{M_z^2}{m_1 x^3} - m_2 g\]

Multiply this by $\dot{x}$ and integrate to obtain
\[\frac{(m_1 + m_2)\dot{x}^2}{2} + \frac{M_z^2}{2m_1 x^2} + m_2 g x = E = \text{const}\]

This is energy conservation: kinetic + rotational + gravitational=total energy.

4. Goldstein Ch. 1, problem 23.

Read Goldstein Section 1.5. In this case $L = m\dot{x}^2/2 + mgx$ and the dissipation function $F = k\dot{x}^2/2$. Eq. (1.70) of Goldstein now reads
\[m\ddot{x} = mg - k\dot{x}\]

In terms of velocity
\[m\dot{v} = mg - kv\]

with the solution
\[v(t) = \left(v(0) - \frac{mg}{k}\right) \exp(-kt/m) + \frac{mg}{k}\]

For a fall from rest
\[v(t) = \frac{mg}{k} \left[1 - \exp(-kt/m)\right]\]

We see that the maximum possible velocity is achieved at large times, as $t \to \infty$, and $v_{\text{max}} = mg/k$. 