Lecture 8

The Kepler problem

$$f = -\frac{k}{r^2} \quad \Rightarrow \quad V = -\frac{k}{r}, \qquad k>0$$

The "θ-u" equation becomes

const determined by initial conditions

$$\theta = \theta' - \int \frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2mku}{\ell^2} - u^2}}$$

Recall that

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \cos^{-1}\left(-\frac{\beta + 2\gamma x}{\sqrt{q}}\right),$$

where

$$q = \beta^2 - 4\alpha\gamma.$$

Here,

$$\begin{cases} \alpha = \dfrac{2mE}{\ell^2}, \\ \beta = \dfrac{2mk}{\ell^2}, \\ \gamma = -1 \end{cases} \Rightarrow q = \left(\frac{2mk}{\ell^2}\right)^2 + 4\,\frac{2mE}{\ell^2} = \underbrace{\left(\frac{2mk}{\ell^2}\right)^2}_{\beta^2}\left[1 + \frac{2E\ell^2}{mk^2}\right].$$

So,

$$\theta = \theta' - \cos^{-1}\left[\frac{\frac{u\ell^2}{mk} - 1}{\sqrt{1 + \frac{2E\ell^2}{mk^2}}}\right]$$

$$\frac{2\gamma u}{\beta} = -2u\,\frac{\ell^2}{2mk} = -\frac{u\ell^2}{mk}$$

Finally,

$$u = \frac{1}{r} = \frac{mk}{\ell^2}\left[1+\sqrt{1+\frac{2E\ell^2}{mk^2}}\cos(\theta-\theta') \right] . \qquad (*)$$

Clearly, $\frac{1}{r}$ is at min (r at max)

when $\theta-\theta' = \pi, 3\pi, \ldots$

$\frac{1}{r}$ is at max (r at min)

when $\theta-\theta' = 0, 2\pi, \ldots$

So θ' is the "turning" angle at which r reaches its min.

Note that the orbit equation $(*)$ depends on 3 constants: $\{E, \ell^2, \theta'\}$. The 4th constant (r_0 or θ_0 indicating the initial position of the particle) does not appear.

Eq. $(*)$ has the form:

$$\frac{1}{r} = C\left[1+\underset{\substack{\uparrow \\ \text{eccentricity}}}{e}\cos(\theta-\theta')\right] \quad \Leftarrow \text{conic eq'n with one focus at the origin}$$

Here, $e = \sqrt{1+\frac{2E\ell^2}{mk^2}}$.

$E>0 \Rightarrow e>1$, hyperbola

$E=0 \Rightarrow e=1$, parabola

$-\frac{mk^2}{2\ell^2}<E<0 \Rightarrow 0<e<1$, ellipse

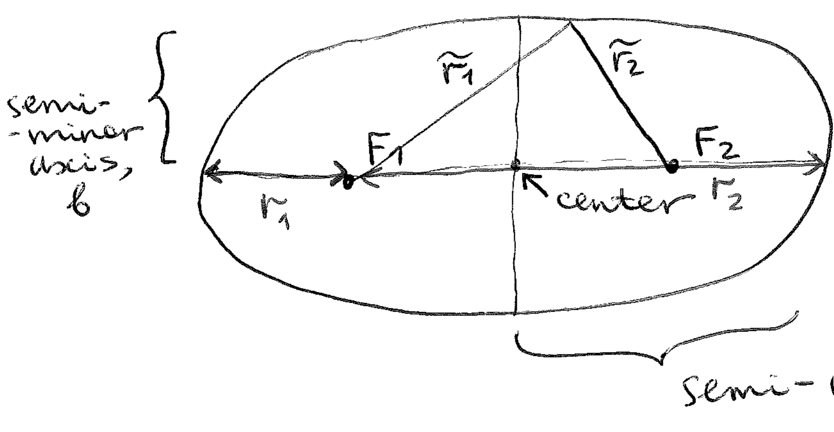
$E=-\frac{mk^2}{2\ell^2} \Rightarrow e=0$, circle $\left[\frac{1}{r}=C_1=\text{const}\right]$

Recall that for a circle, we had

$$E=V'(r_0)=\underbrace{V(r_0)}_{-\frac{k}{r_0}}+\frac{\ell^2}{2mr_0^2}\underset{\substack{\uparrow \\ r_0=\frac{\ell^2}{mk}}}{=}-\frac{mk^2}{\ell^2}+\underbrace{\frac{\ell^2}{2m}\frac{m^2k^2}{\ell^4}}_{\frac{mk^2}{2\ell^2}} = $$

$$= -\frac{mk^2}{2\ell^2}$$, consistent with above.

Now, consider elliptic orbits:



$\tilde{r}_1+\tilde{r}_2=\text{const}$ for all points on the curve

$\begin{cases} r_1 = \text{min distance wrt } F_1 \\ r_2 = \text{max distance wrt } F_1 \end{cases} \Leftarrow$ "apsidal distances"

Clearly, $a=\frac{r_1+r_2}{2}$.

Moreover, r_1 & r_2 are the turning points at which $T=0$.

But then

$$E = \frac{\ell^2}{2mr^2} + \underbrace{V(r)}_{-\frac{k}{r}}\Big|_{r=r_1,r_2} \quad , \text{ or}$$

$$\frac{r^2}{E} * \Big| \quad E - \frac{\ell^2}{2mr^2} + \frac{k}{r} = 0 \, ,$$

$$r^2 + \frac{k}{E} r - \frac{\ell^2}{2mE} = 0 \quad \Leftarrow r_{1,2} \text{ are square roots of this quadratic eq'n}$$

Now, $\frac{r_1 + r_2}{2} = -\frac{k}{2E} = a$

a depends solely on E (not on ℓ)

So, $E = -\frac{k}{2a}$. $\quad (**)$

For a circle, $E = -\frac{mk^2}{2\ell^2} = -\frac{k}{2\underbrace{r_0}_{\frac{\ell^2}{mk}}}$, consistent with $(**)$.

Next,

$$e = \sqrt{1 + \frac{2E\ell^2}{mk^2}} = \sqrt{1 - \frac{\ell^2}{mka}} \quad , \text{ or}$$

$$\frac{\ell^2}{mk} = a(1 - e^2) .$$

The orbit eq'n becomes:

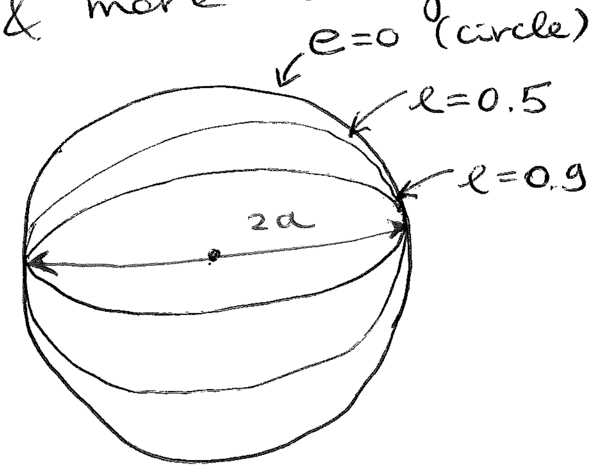
$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\theta')}.$$

In this eq'n, r_1 (min dist.) corresponds to $\theta-\theta'=0$ & r_2 (max dist.) to $\theta-\theta'=\pi$.

Then
$$\begin{cases} r_1 = \dfrac{a(1-e^2)}{1+e} = a(1-e), \\ r_2 = \dfrac{a(1-e^2)}{1-e} = a(1+e). \end{cases}$$

If we fix a $\Rightarrow$ E is fixed as well.

But e can vary depending on l, resulting in more & more elongated orbits:



Kepler problem: motion in time

Recall that

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}}$$

$$\Downarrow$$

$$t = \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{dr}{\sqrt{E + \frac{k}{r} - \frac{\ell^2}{2mr^2}}} \quad (*) \qquad \text{get } t(r)$$

Next, recall that $dt = \frac{mr^2}{\ell} d\theta$

Since $r(\theta) = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta')}$, we obtain:

use $a(1-e^2) = \frac{\ell^2}{mk}$ below

$$t = \underbrace{\frac{m}{\ell}\left(\frac{\ell^2}{mk}\right)^2}_{\frac{\ell^3}{mk^2}} \int_{\theta_0}^{\theta} d\theta \frac{1}{\left(1 + e\cos(\theta - \theta')\right)^2} \quad (**) \qquad \text{get } t(\theta)$$

Consider $e=1$ (parabola) for simplicity. Also, set $\theta'=0$ (i.e., measure all angles from the angle at which $r=r_1$: min dist., or perihelion).

Then Eq. (**) becomes:

$$t = \frac{\ell^3}{4mk^2}\int_0^{\theta}\frac{d\theta}{\cos^4\left(\frac{\theta}{2}\right)} = \frac{\ell^3}{2mk^2}\left[\tan\left(\frac{\theta}{2}\right)+\frac{1}{3}\tan^3\left(\frac{\theta}{2}\right)\right]$$

set $\theta_0=0$ as well
(i.e. start from perihelion)

$t=0 \Leftrightarrow \theta=0$, particle at perihelion as expected

$t\to-\infty \Rightarrow \theta=-\pi$ particle approaches from infinity

$t\to+\infty \Rightarrow \theta=+\pi$ particle escapes to infinity

---

For <u>elliptical</u> <u>motion</u>, it is convenient to introduce ψ through

eccentric anomaly

$$r = a(1-e\cos\psi)$$

Recall the orbit eq'n: $r = \dfrac{a(1-e^2)}{1+e\cos\theta}$.

$\begin{cases} \theta=0 \text{ corresponds to the } \underline{\text{min dist.}}\text{: } \psi=0 \text{ as well} \\ \qquad\qquad\qquad\qquad\qquad\quad \text{perihelion} \\ \theta=\pi \text{ corresponds to the } \underline{\text{max dist.}}\text{: } \psi=\pi \text{ as well} \\ \qquad\qquad\qquad\qquad\qquad\quad \text{aphelion} \end{cases}$

Clearly, as θ goes from 0 to 2π, so does ψ.

$$\text{Now,}\quad t = \overset{\text{Eq. }(*)}{\sqrt{\frac{m}{2}}} \int_{r_0}^{r} \frac{r\,dr}{\sqrt{Er^2 + kr - \frac{\ell^2}{2m}}} \;\textcircled{=}$$

$$\begin{cases} \dfrac{\ell^2}{mk} = a(1-e^2), \\ E = -\dfrac{k}{2a} \end{cases}$$

$$\textcircled{=} \sqrt{\frac{m}{2k}} \int_{r_0}^{r} \frac{r\,dr}{\sqrt{r - \frac{r^2}{2a} - \frac{a(1-e^2)}{2}}} =$$

r_0: perihelion distance (prev. called r_1)

$$= \sqrt{\frac{m}{2k}} \int_0^{\psi} \frac{a(1-e\cos\psi)\, ae\sin\psi\, d\psi}{\sqrt{a(1-e\cos\psi) - \frac{a^2(1-e\cos\psi)^2}{2a} - \frac{a(1-e^2)}{2}}} \;\diamond\!\!=$$

$$\begin{cases} r = a(1-e\cos\psi), \\ dr = ae\sin\psi\, d\psi \end{cases}$$

$$\diamond\!\!= \sqrt{\frac{ma^3}{2k}} \int_0^{\psi} d\psi \frac{e\sin\psi(1-e\cos\psi)}{\sqrt{-\cancel{e\cos\psi} + \cancel{e\cos\psi} - \frac{1}{2}e^2\cos^2\psi + \frac{e^2}{2}}} =$$

$$= \sqrt{\frac{ma^3}{2k}} \int_0^{\psi} d\psi \frac{e\sin\psi(1-e\cos\psi)}{\frac{e}{\sqrt{2}}\underbrace{\sqrt{1-\cos^2\psi}}_{\sin\psi}} = \sqrt{\frac{ma^3}{k}} \int_0^{\psi} d\psi'(1-e\cos\psi')$$

$$= \sqrt{\frac{ma^3}{k}}\,[\psi - e\sin\psi].$$

Note that integrating $\int_0^{2\pi}$ will give τ, the period of the elliptical motion:

$$\tau = \sqrt{\frac{ma^3}{k}} \int_0^{2\pi} d\psi \, (1 - e\cos\psi) = \sqrt{\frac{ma^3}{k}}\, 2\pi .$$

This result can be found more directly by considering angular momentum:

Consider $\underbrace{\frac{dA}{dt}}_{\text{areal velocity}} = \frac{1}{2} r^2 \dot{\theta} = \frac{\ell}{2m}$.

Then $\int_0^{\tau} dt \, \frac{dA}{dt} = A = \frac{\ell \tau}{2m}$.

For an ellipse, $\begin{cases} A = \pi a b, \\ b = a\sqrt{1-e^2} \end{cases}$

$\Downarrow$

$$b = a\sqrt{1 - 1 + \frac{\ell^2}{mka}} = \sqrt{\frac{\ell^2 a}{mk}} .$$

Finally, $\tau = \frac{2m}{\ell} A = \frac{2m}{\ell} \pi a^{\frac{3}{2}} \sqrt{\frac{\ell^2}{mk}} =$

$= 2\pi a^{3/2} \sqrt{\frac{m}{k}}$, same as <u>above</u>.

Note that $\tau^2 \sim a^3$ [3rd Kepler's law]
~1610

Recall however that planet motion about the Sun is a two-body problem, with

$$\mu = \frac{\overset{\text{planet}}{m_1}\,\overset{\text{Sun}}{m_2}}{m_1+m_2} \quad \& \quad f = -G\,\frac{m_1 m_2}{r^2}$$

[i.e., $k = G m_1 m_2$]

Then $\tau = \dfrac{2\pi a^{3/2}}{\sqrt{G m_1 m_2}}\sqrt{\dfrac{m_1 m_2}{m_1+m_2}} = \dfrac{2\pi a^{3/2}}{\sqrt{G(m_1+m_2)}} \approx$

$\approx \dfrac{2\pi a^3}{\sqrt{G m_2}}$ if $m_2 \gg m_1$.

So the τ^2 vs. a^3 for all planets will not be a straight line exactly.

Next, consider

$$\omega = \frac{2\pi}{\tau} = \sqrt{\frac{k}{m a^3}}.$$

transcendental eq'n, generally solved numerically

Then $\underbrace{\omega t}_{\text{goes from } 0 \text{ to } 2\pi \text{ along with } \theta \,\&\, \psi} = \psi - e \sin\psi$ (*) $\Leftarrow$ Kepler's eq'n, relates ψ & t

We need to invert Eq. (*) above to obtain $r(t)$; for $\theta(t)$, we first need to relate θ and ψ:

$$\underbrace{\frac{a(1-e^2)}{1+e\cos\theta} = \underbrace{a(1-e\cos\psi)}_{r}}_{\text{orbit eq'n}} \quad \text{, or}$$

$$1+e\cos\theta = \frac{1-e^2}{1-e\cos\psi} \,,$$

$$\cos\theta = e^{-1}\,\frac{\cancel{1}-e^2-\cancel{1}+e\cos\psi}{1-e\cos\psi} = \frac{\cos\psi - e}{1-e\cos\psi}\,.$$

Now,
$$\begin{cases} 1-\cos\theta = \dfrac{1-e\cos\psi-\cos\psi+e}{1-e\cos\psi} = \dfrac{(1+e)(1-\cos\psi)}{1-e\cos\psi} \\ \% \\ 1+\cos\theta = \dfrac{1-e\cos\psi+\cos\psi-e}{1-e\cos\psi} = \dfrac{(1-e)(1+\cos\psi)}{1-e\cos\psi} \end{cases}$$

$$\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \frac{1+e}{1-e}\,\frac{\sin^2\frac{\psi}{2}}{\cos^2\frac{\psi}{2}} \quad \text{, or}$$

$$\underbrace{\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{\psi}{2}}_{\text{relates } \theta \,\&\, \psi}$$