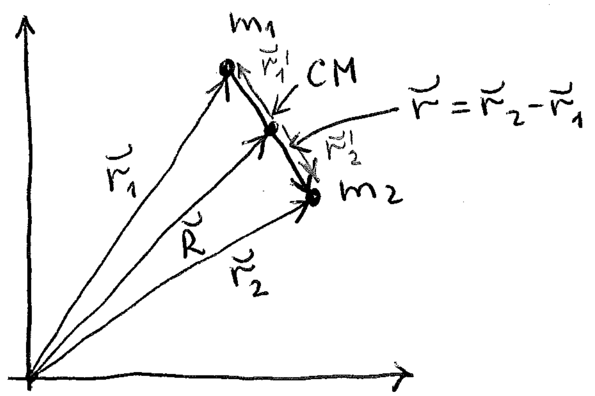
Lecture 6

The central force problem

Consider 2 point particles interacting via $U = U(\vec r, \dot{\vec r}, \dots)$, where $\vec r = \vec r_2 - \vec r_1$:



$$\vec R = \frac{m_1 \vec r_1 + m_2 \vec r_2}{m_1 + m_2}$$

$$\mathcal{L} = T(\dot{\vec R}, \dot{\vec r}) - U(\vec r, \dot{\vec r}, \dots)$$

Introduce $\begin{cases} \vec R + \vec r_1' = \vec r_1 , \\ \vec R + \vec r_2' = \vec r_2 \end{cases} \Rightarrow \vec r_1' = \vec r_1 - \dfrac{m_1 \vec r_1 + m_2 \vec r_2}{m_1 + m_2} = -\dfrac{m_2}{m_1 + m_2} \vec r$.

Likewise, $\vec r_2' = \vec r_2 - \vec R = \dfrac{m_1}{m_1 + m_2} \vec r$

Now,
$$T = \frac{m_1}{2} \dot{\vec r}_1^{\,2} + \frac{m_2}{2} \dot{\vec r}_2^{\,2} = \underbrace{\frac{m_1 + m_2}{2} \dot{\vec R}^2}_{\text{CoM motion}} + \underbrace{\frac{m_1}{2} \dot{\vec r}_1'^{\,2} + \frac{m_2}{2} \dot{\vec r}_2'^{\,2}}_{\text{relative motion}} .$$

Indeed,

$$T = \frac{m_1 + m_2}{2 (m_1 + m_2)^2} (m_1 \dot{\vec r}_1 + m_2 \dot{\vec r}_2)^2 + \frac{m_1 m_2^2}{2 (m_1 + m_2)^2} (\dot{\vec r}_2 - \dot{\vec r}_1)^2 + \frac{m_2 m_1^2}{2 (m_1 + m_2)^2} (\dot{\vec r}_2 - \dot{\vec r}_1)^2 \quad \boxed{=}$$

$\dot{\vec{r}}_1^{\,2}$ prefactor:

$$\frac{m_1^2(m_1+m_2) + \overbrace{m_1 m_2^2 + m_2 m_1^2}^{m_1 m_2(m_1+m_2)}}{2(m_1+m_2)^2} = \frac{m_1^2 + m_1 m_2}{2(m_1+m_2)} = \frac{m_1}{2}$$

$\dot{\vec{r}}_2^{\,2}$ prefactor:

$$\frac{m_2^2(m_1+m_2) + m_1 m_2(m_1+m_2)}{2(m_1+m_2)^2} = \frac{m_2}{2}$$

$\dot{\vec{r}}_1 \dot{\vec{r}}_2$ prefactor:

$$\frac{(m_1+m_2)m_1 m_2 - m_1 m_2^2 - m_1^2 m_2}{(m_1+m_2)^2} = \emptyset$$

$\boxed{=}$ $\frac{m_1}{2}\dot{\vec{r}}_1^{\,2} + \frac{m_2}{2}\dot{\vec{r}}_2^{\,2}$, as expected.

Now, $\mathcal{L} = \frac{m_1+m_2}{2}\dot{\vec{R}}^2 + \frac{m_1 m_2^2}{2(m_1+m_2)^2}\dot{\vec{r}}^2 + \frac{m_2 m_1^2}{2(m_1+m_2)^2}\dot{\vec{r}}^2 -$

$- \mathcal{U}(\vec{r}, \dot{\vec{r}}, \dots) = \frac{m_1+m_2}{2}\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2 - \mathcal{U}(\vec{r}, \dot{\vec{r}}, \dots)$

$\vec{R}$ is cyclic $\Rightarrow$ CoM is either at rest or moves uniformly $\Rightarrow \dot{\vec{R}} = \text{const}$

Thus the 1st term can be dropped from $\mathcal{L}$:

$$\mathcal{L} = \frac{\mu}{2}\dot{\vec{r}}^2 - \mathcal{U}(\vec{r}, \dot{\vec{r}}, \dots)$$

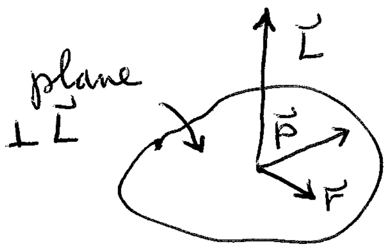
↑ single particle at distance $\vec{r}$ from the origin, with reduced mass $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$.

Now, consider $V = V(r)$, where $r = |\vec r|$.

The problem has spherical symmetry: rotation about any axis leaves the system inv.

But then $\vec L = \vec r \times \vec p$ is conserved

If $\vec L \neq 0$, $\vec r$ & $\vec p$ lie in a plane $\perp \vec L$:



If $\vec L = 0$, $\vec r \uparrow\uparrow \vec p \Rightarrow$ motion in a straight line

Thus motion is always in a plane (and may be along a line); can be described using polar coords $\{r, \theta\}$:

$$\mathcal{L} = T - V = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2) - V(r)$$

θ is a cyclic coord (as expected given the spherical symm.), and

$|\vec L|$, magnitude of angular momentum

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = m r^2 \dot\theta = \underbrace{\ell}_{\text{const}}$$

$$\Uparrow$$

$$\frac{d}{dt}(m r^2 \dot\theta) = 0$$

Further, $\dfrac{d}{dt}\Big(\underbrace{\tfrac{1}{2}r^2\dot{\theta}}\Big) = 0 \Rightarrow \dot{A} = \text{const}$

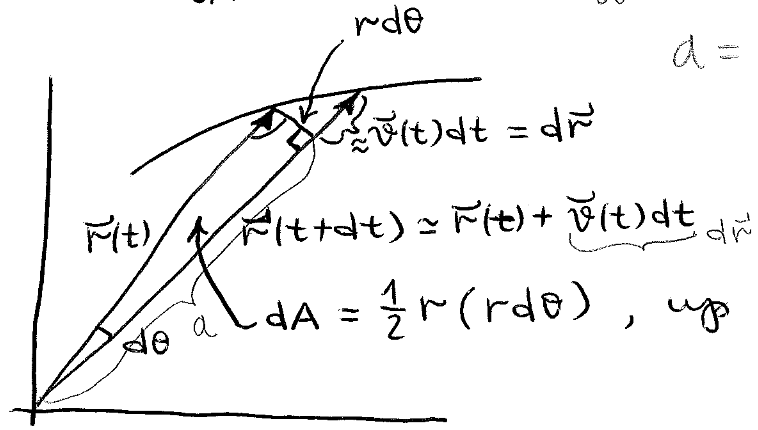
↑ Kepler's 2nd law of planetary motion

$\dot{A}$, areal velocity (area swept by $\vec{r}$ per unit time)

Indeed, $\underbrace{dA}_{\text{area swept in } dt} = \frac{1}{2} r (r\,d\theta) \Rightarrow \dot{A} = \frac{r^2}{2}\dot{\theta}$

$d^2 = r^2 - r^2 d\theta^2$ gives

$d = r\underbrace{\sqrt{1-d\theta^2}}_{\simeq 1 - \frac{d\theta^2}{2}} \simeq r$



$dA = \frac{1}{2} r (r\,d\theta)$, up to an $O(d\theta |d\vec{r}|)$ correction

The radial LE is given by

$$\frac{d}{dt}(m\dot{r}) - m r \dot{\theta}^2 + \underbrace{\frac{\partial V}{\partial r}}_{-f(r)} = 0$$

$-f(r)$, where $f(r)$ is a force along $\vec{r}$

So, $m\ddot{r} - m r \dfrac{\ell^2}{m^2 r^4} = f(r)$, or

$$m\ddot{r} - \frac{\ell^2}{m r^3} = f(r).$$

Forces are conservative:

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \text{const}$$

↑ total energy

This can also be seen in EoM:

$$\dot r \times \Big| \quad m\ddot r = -\frac{d}{dr}\left(V(r) + \frac{1}{2}\frac{\ell^2}{mr^2}\right)$$

Multiply both sides by $\dot r$ and note that

$$\begin{cases} m\ddot r \dot r = \dfrac{d}{dt}\left(\dfrac{m\dot r^2}{2}\right), \\ \dfrac{d}{dt} g(r(t)) = \dfrac{dg}{dr}\dot r \quad \text{for any f'n } g(r) \end{cases}$$

Hence $\quad \dfrac{d}{dt}\left(\dfrac{m\dot r^2}{2}\right) = -\dfrac{d}{dt}\left(V + \dfrac{\ell^2}{2mr^2}\right)$, or

$$\frac{m}{2}\dot r^2 + \underbrace{\frac{\ell^2}{2mr^2}}_{\frac{1}{2mr^2} m^2 r^4 \dot\theta^2 = \frac{mr^2\dot\theta^2}{2}} + V = \text{const} = E \text{, total energy}$$

But then $\quad \dot r = \sqrt{\dfrac{2}{m}\left(E - V - \dfrac{\ell^2}{2mr^2}\right)}$, or

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}},$$

$$t_{(r)} = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}} \qquad (*)$$

initial value of r at time 0

(*) can be solved and inverted to obtain $r(t)$ in terms of E, ℓ, r_0.

Finally, $d\theta = \dfrac{\ell\, dt}{mr^2}$, yielding

$$\theta(t) = \ell \int_0^t \frac{dt'}{mr^2(t')} + \theta_0 .$$

Thus $\theta(t)$ is expressed in terms of

$\underbrace{E, \ell, r_0, \theta_0}_{4}$. const of integration

Equivalent 1D problem & classification of orbits

Note that EoM $m\ddot{r} = \overbrace{f(r) + \dfrac{\ell^2}{mr^3}}^{\equiv f'(r)}$ obtained above involves only r & its derivatives, so the problem is 1D.

Recall that $\dfrac{\ell^2}{mr^3} = \dfrac{m^2 r^4 \dot{\theta}^2}{mr^3} = mr\dot{\theta}^2 = \dfrac{m v_\theta^2}{r}$, $v_\theta = r\dot{\theta}$, so the extra term is just the centrifugal force.

Further, $f' = -\dfrac{\partial V'}{\partial r} \Rightarrow V' = V + \dfrac{\ell^2}{2mr^2}$.

The energy conservation is then given by

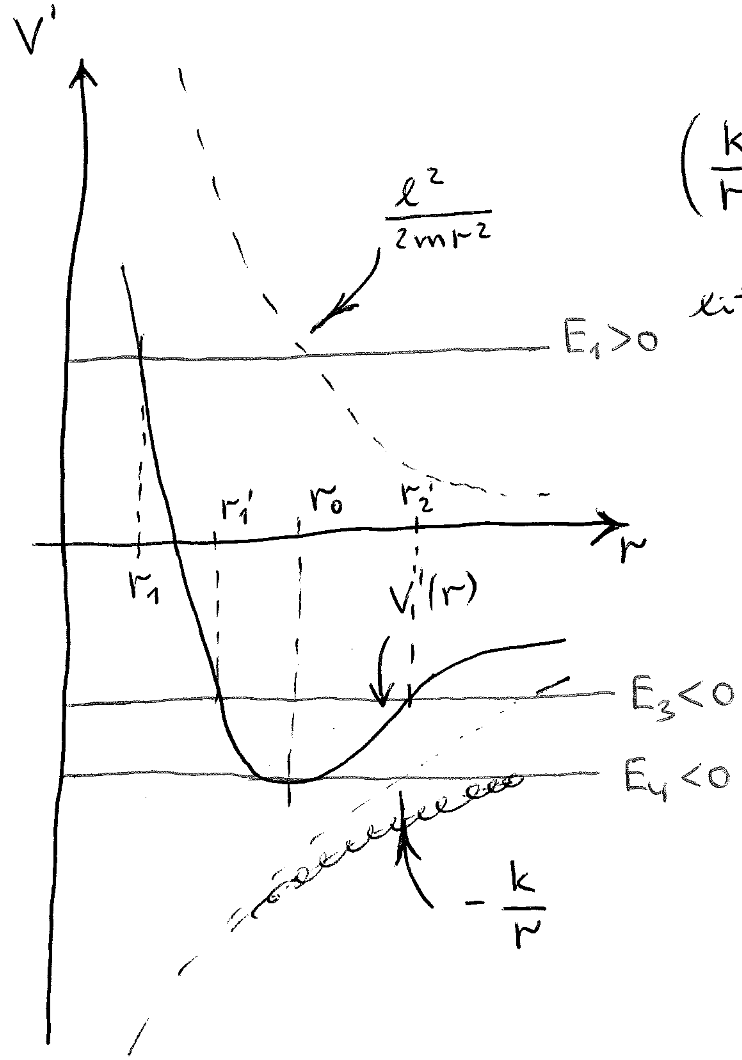
$$E = V'(r) + \frac{1}{2} m \dot{r}^2$$

as an example, focus on

$$f = -\frac{k}{r^2} \Rightarrow V = -\frac{k}{r},$$

$(k>0)$

$$V'_{(r)} = -\frac{k}{r} + \frac{\ell^2}{2mr^2}.$$

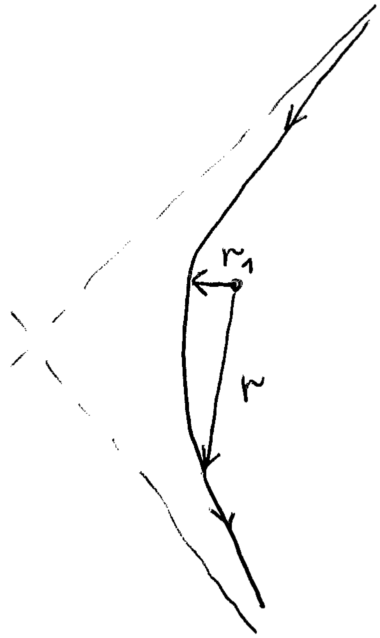


$$\left.\left(\frac{k}{r^2} - \frac{\ell^2}{mr^3}\right)\right|_{r=r_0} = 0$$ gives either $r_0 = \infty$ or

$$r_0 = \frac{\ell^2}{mk}.$$

Consider a particle with energy E_1, as shown in the Fig. Clearly, at $r=r_1$ $E_1 = V'(r_1)$ & thus $T=0$: the particle cannot enter the $r<r_1$ region.

The particle "comes in", strikes the "effective repulsive barrier", and travels back out to infinity:



Note that

$$E - V' = \frac{1}{2} m \dot{r}^2 \quad , \quad r \geq r_1$$

$$[\text{ if } r = r_1 \; , \; \dot{r}\big|_{r=r_1} = 0 \;]$$

at the same time,

$$V' - V = \frac{\ell^2}{2mr^2} = \frac{m^2 r^4 \dot{\theta}^2}{2mr^2} = \frac{1}{2} m r^2 \dot{\theta}^2 .$$

So we can extract both $v_r = \dot{r}$ & $v_\theta = r\dot{\theta}$ as a f'n of r from these curves.

Consider now $E_3 < 0$ (see Fig. above). There're now 2 turning points, r_1' & r_2'. So the orbits are bounded between r_1' & r_2' but not necessarily closed (see Fig. 3.7 in the book).

If now $E_4 < E_3$ s.t. $V'(r_0) = E_4$, we must have $\dot{r} = 0$ at all times.

But then the orbit is a circle ($r = \text{const}$). Clearly, $f'(r_0) = 0$, or
r_0"

$$\underbrace{f(r_0) = -\frac{\ell^2}{m r_0^3} = -m r_0 \dot{\theta}^2}_{\text{applied external force is equal \& opposite to the "centripetal" force}}$$

Motion with $E < E_4$ is impossible with this potential.

The classification into open, bounded, an circular orbits remains true for any potential that

(i) falls off slower than $\frac{1}{r^2}$ as $r \to \infty$ (so that it predominates over the $\sim \frac{1}{r^2}$ centrifugal term) at large r

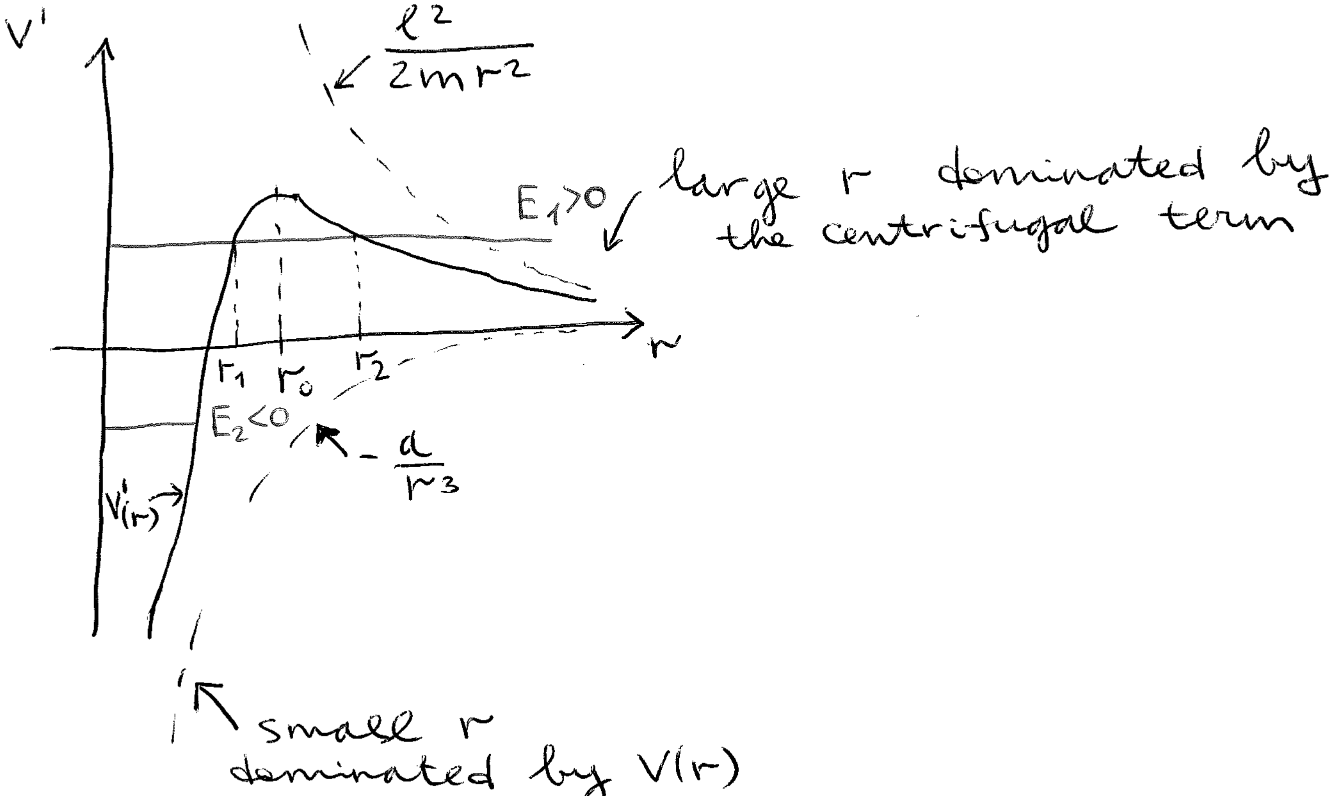
(ii) becomes infinite slower than $\frac{1}{r^2}$ as $r \to 0$ (so that the centrifugal term predominates at small r)

So, for $V(r) = -\frac{k}{r^p}$, $1 \le p < 2$ is OK.

However, other cases can be treated too: for ex., consider

$V(r) = -\frac{a}{r^3}$ which breaks the rule above

In this case, $V'(r)$ looks like this:



For E_1 shown in the Fig., the motion is either bounded $(0 \le r \le r_1)$ or unbounded $(r \geq r_2)$ depending on the initial conditions.
The $r_1 < r_0 < r_2$ region is not accessible with this value of E_1.

For $E_2 < 0$, only bounded motion is possible.

Other interesting cases can be considered, such as $V = \frac{1}{2} k r^2$ (isotropic harmonic oscillator).