Finite canonical transformations } Lecture 24

Suppose that the trajectory in phase space is parameterized by α, with the initial condition $\alpha = 0$. Then if u is some f'n of the system configuration [but not of time], we have $u = u(\alpha)$, with init. value $u_0 = u(0)$.

Then $\underbrace{\delta u}_{\text{infinitesimal change along the trajectory}} = d\alpha\,[u, G]$, or $\frac{du}{d\alpha} = [u, G]$. (*)

We can integrate Eq. (*) to obtain $u(\alpha)$:

consider $u(\alpha) = u_0 + \alpha \left.\frac{du}{d\alpha}\right|_{\alpha=0} + \frac{\alpha^2}{2!} \left.\frac{d^2u}{d\alpha^2}\right|_{\alpha=0} + \ldots$

Taylor expansion

But $\left.\frac{du}{d\alpha}\right|_{\alpha=0} = [u, G]|_{\alpha=0}$.

Next, $\left.\frac{d^2u}{d\alpha^2}\right|_{\alpha=0} = \frac{d}{d\alpha}[u, G]\Big|_0 = [[u, G], G]\Big|_0$, etc.

Hence

$$u(\alpha) = u_0 + \alpha\,[u, G]\Big|_0 + \frac{\alpha^2}{2!}[[u, G], G]\Big|_0 + \ldots \qquad (**)$$

If $u \to \xi_i$ & $u_0 \to \eta_i$ $(\forall i)$, then Eq. (**) is a finite canonical transform'n $\eta_i \to \xi_i$ generated by G.

Ex.

1. Consider $G = L_z = X_i P_{i,y} - Y_i P_{i,x}$.

i labels a particle

Then $$[X_i, L_z] = \underbrace{\frac{\partial X_i}{\partial X_j}}_{=\delta_{ij}} \frac{\partial L_z}{\partial P_{j,x}} + \underbrace{\frac{\partial X_i}{\partial Y_j}}_{=0} \frac{\partial L_z}{\partial P_{j,y}} - \underbrace{\frac{\partial X_i}{\partial P_{j,x}}}_{=0} \frac{\partial L_z}{\partial X_j} - \underbrace{\frac{\partial X_i}{\partial P_{j,y}}}_{=0} \frac{\partial L_z}{\partial Y_j} = -Y_i \quad , \text{ and}$$

$$[Y_i, L_z] = \underbrace{\frac{\partial Y_i}{\partial Y_j}}_{=\delta_{ij}} \frac{\partial L_z}{\partial P_{j,y}} = X_i \quad .$$

In this case, α is the rot'n angle around z-axis. To compute the change in X_i under the finite rot'n, consider

$$\begin{cases} [X_i, L_z]\big|_0 = -y_i , \\ [[X_i, L_z], L_z]\big|_0 = -x_i , \\ [[[X_i, L_z], L_z], L_z]\big|_0 = y_i , \text{ etc.} \end{cases}$$

Then Eq. (**) gives:

$$X_i = x_i - \alpha y_i - \frac{\alpha^2}{2!} x_i + \frac{\alpha^3}{3!} y_i + \frac{\alpha^4}{4!} x_i + \ldots =$$

$$= x_i \left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \ldots \right) - y_i \left(\alpha - \frac{\alpha^3}{3!} + \ldots \right) =$$

$$= x_i \cos\alpha - y_i \sin\alpha \quad , \text{ as expected.}$$

Similarly, $\begin{cases} [Y_i, L_z]\big|_0 = x_i , \\ [[Y_i, L_z], L_z]\big|_0 = -y_i , \\ [[[Y_i, L_z], L_z], L_z]\big|_0 = -x_i , \text{ etc.} \end{cases}$

$$Y_i = y_i + \alpha x_i - \frac{\alpha^2}{2!} y_i - \frac{\alpha^3}{3!} x_i + \ldots =$$

$$= x_i \left(\alpha - \frac{\alpha^3}{3!} + \ldots\right) + y_i \left(1 - \frac{\alpha^2}{2!} + \ldots\right) =$$

$$= x_i \sin\alpha + y_i \cos\alpha ,$$ again as expected.

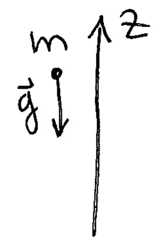
2. Consider $G = H$ and $\alpha = t$, then

Eq. (*) gives $\frac{du}{dt} = [u, G] \Leftarrow$ EoM for u

Formal solution (Eq. (**)):

$$u(t) = \underbrace{u_0}_{u(t=0)} + t[u, H]\big|_0 + \frac{t^2}{2!}[[u, H], H]\big|_0 + \ldots$$

Now, consider $H = \frac{p_z^2}{2m} + mgz$.



$$[z, H] = \frac{p_z}{m} ,$$

$$[[z, H], H] = \frac{1}{m}[p_z, H] = -\frac{1}{m}\frac{\partial p_z}{\partial p_z}\frac{\partial H}{\partial z} = \underbrace{-g}_{\text{const}} .$$

all higher-order brackets vanish and

Eq. (**) gives:

$$z = z_0 + \underbrace{\frac{p_{z,0}}{m} t}_{v_{z,0} t} - \frac{gt^2}{2}$$

Finally, define $\hat{G}u = [u, G]$, then

Eq. (**) becomes $\hat{G}(\hat{G}u) = \hat{G}^2 u = [[u, G], G]$, etc.

$$u(\alpha) = u_0 + \alpha(\hat{G}u)\Big|_0 + \frac{\alpha^2}{2!}(\hat{G}^2 u)\Big|_0 + \ldots =$$

$$= \underline{\underline{e^{\hat{G}\alpha} u\Big|_0}}\,.$$

formal solution, to be understood as an infinite series

For ex., $u(t) = e^{\hat{H}t} u\Big|_0$ is the formal sol'n to EoMs

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<u>Angular momentum PBs</u>

Consider $u \to F_i$ (component of $\vec{F}$ in a fixed frame):

$$\partial F_i = d\alpha\,[F_i, G]$$

If we focus on a rot'n of the system around $\underbrace{\vec{n}}_{\text{fixed axis}}$, $G = \vec{L}\cdot\vec{n}$ and

$$\partial \vec{F} = d\alpha\,[\vec{F}, \underbrace{\vec{L}\cdot\vec{n}}_{\text{generator of a spatial rot'n of the system vars }(q,p)}]$$

Clearly, we have to have $\vec{F} = \vec{F}(q,p)$ for $\vec{F}$ to be rotated by an ICT.

But then $d\vec{F} = \underbrace{d\alpha}_{\text{infinitesimal change in angle}}\,\vec{n}\times\vec{F}$, and we have:

$$d\alpha\,[\vec{F}, \vec{L}\cdot\vec{n}] = d\alpha\, \vec{n}\times\vec{F}, \text{ or}$$

$$\underline{\underline{[\vec{F}, \vec{L}\cdot\vec{n}] = \vec{n}\times\vec{F}.}} \qquad (***)$$

<u>Ex.</u> a particle with momentum $\vec{p}$ in 3D Cartesian space.

$$\vec{n} = \hat{z} \Rightarrow \vec{L}\cdot\vec{n} = L_z = x p_y - y p_x .$$

Then
$$\begin{cases} [p_x, \underbrace{x p_y - y p_x}_{L_z}] = -\dfrac{\partial p_x}{\partial p_x}\dfrac{\partial L_z}{\partial x} = -p_y, \\ [p_y, L_z] = -\dfrac{\partial p_y}{\partial p_y}\dfrac{\partial L_z}{\partial y} = p_x, \\ [p_z, L_z] = 0. \end{cases}$$

Clearly, $\hat{z}\times\vec{p} = \begin{pmatrix} -p_y \\ p_x \\ 0 \end{pmatrix}$, just as $(***)$ predicts

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Eq. $(***)$ can be expressed as

$$[F_i, L_j n_j] = e_{ijk} n_j F_k, \text{ which implies}$$

$$\underline{\underline{[F_i, L_j] = e_{ijk} F_k.}}$$

Further, $[\underbrace{\vec{F}\cdot\vec{G}}_{\text{scalar}}, \vec{L}\cdot\vec{n}] = F_i [G_i, L_j n_j] +$

$$+ G_i [F_i, L_j n_j] =$$

$$= F_i e_{ijk} n_j G_k + G_i \underset{i\leftrightarrow k}{e_{ijk} n_j F_k} \circleddash$$

$\Leftrightarrow (e_{ijk} + e_{ikj}) F_i n_j G_k = 0$, as expected (scalars are inv under a rot'n)

Clearly, $[\overbrace{|\vec{F}|^2}^{F^2}, \vec{L}\cdot\vec{n}] = 0 \Rightarrow$ the magnitude of any vector has a vanishing PB with L_i, $\forall i = \{x, y, z\}$.

If $\vec{F} = \vec{L}$, we have:

$$[\vec{L}, \vec{L}\cdot\vec{n}] = \vec{n}\times\vec{L}, \text{ or}$$

$$[L_i, L_j] = e_{ijk} L_k .$$

[PB of canonical momenta must always $= 0$. Therefore, no two components of $\vec{L}$ can be used simultaneously as canonical vars. However, any component of $\vec{L}$ & L^2 can be.]

Next, $[L^2, \vec{L}\cdot\vec{n}] = 0$.

If L_x & L_y are constants of the motion, $[L_x, L_y] = L_z$ is also a const of the motion by Poisson's theorem. Then $\vec{L} = (L_x, L_y, L_z)$ is conserved.

Moreover, $[\vec{p}, \vec{L}\cdot\vec{n}] = \vec{n}\times\vec{p} \Rightarrow [p_i, L_j] = e_{ijk} p_k$.

If, in addition, $p_z = \text{const}$, we have:

$$\begin{cases} [p_z, L_x] = p_y, \\ [p_z, L_y] = -p_x \end{cases} \Rightarrow \text{both } \vec{L} \text{ and } \vec{p} = (p_x, p_y, p_z) \text{ are conserved}$$