Variational principle ~~of~~ and Hamilton's equations

Lecture 20

Recall that $\delta \int_{t_1}^{t_2} dt\, \mathcal{L} = 0$ leads to Euler-Lagrange's EoM. Can we do the same for Hamilton's EoM, so that paths are varied in ~~[illegible]~~ phase space rather than configuration space?

Consider
$$\underbrace{\delta \int_{t_1}^{t_2} dt \underbrace{\left(p_i \dot{q}_i - H(p,q,t)\right)}_{= f}}_{\delta I} = 0$$

This problem is of the type

$$\delta \int_{t_1}^{t_2} dt\, f(q, p, \dot{q}, \dot{p}, t) = 0 ,$$ in which p's and q's are treated as indep. vars.

The Euler-Lagrange EoM are:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial f}{\partial \dot{q}_j}\right) - \dfrac{\partial f}{\partial q_j} = 0 , \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial f}{\partial \dot{p}_j}\right) - \dfrac{\partial f}{\partial p_j} = 0 \end{cases} \qquad j = 1, \dots, n$$

In our case, $\dfrac{\partial f}{\partial \dot{q}_j} = p_j$ & $\dfrac{\partial f}{\partial q_j} = -\dfrac{\partial H}{\partial q_j}$,

so that

$$\dot{p}_j + \frac{\partial H}{\partial q_j} = 0 \qquad (*)$$

Moreover, $\frac{\partial f}{\partial \dot{p}_j} = 0$ & $\frac{\partial f}{\partial p_j} = \dot{q}_j - \frac{\partial H}{\partial p_j}$, yielding

$$\dot{q}_j - \frac{\partial H}{\partial p_j} = 0 \qquad (**)$$

(*) & (**) are Hamilton's EoM.

The Lagrangian framework required only $\delta q_i = 0$ at end-points, whereas here it appears that we need both $\delta q_i = 0$ & $\delta p_i = 0$.

$q_i(t,\alpha) = \overbrace{q_i(t,0)}^{\text{real path}} + \alpha\,\eta_i(t)$, $\eta_i(t_1) = \eta_i(t_2) = 0$

However, in this case

$$\frac{\partial I}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} dt \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial q_i}{\partial \alpha} + \frac{\partial f}{\partial p_i} \frac{\partial p_i}{\partial \alpha} + \underbrace{\frac{\partial f}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \alpha} + \frac{\partial f}{\partial \dot{p}_i} \frac{\partial \dot{p}_i}{\partial \alpha}}_{\text{by parts}} \right) d\alpha$$

Note that

$\int_{t_1}^{t_2} dt \frac{\partial f}{\partial \dot{p}_i} \frac{\partial \dot{p}_i}{\partial \alpha} = 0$ identically (f is indep. of $\dot{p}_i$'s), so that there's no need to set $\delta p_i = 0$ at end-points.

In contrast,

$$\int_{t_1}^{t_2} dt\, \frac{\partial f}{\partial \dot q_i}\underbrace{\frac{\partial \dot q_i}{\partial \alpha}}_{\frac{d}{dt}\left(\frac{\partial q_i}{\partial \alpha}\right)} = \left.\frac{\partial f}{\partial \dot q_i}\underbrace{\frac{\partial q_i}{\partial \alpha}}_{\eta_i}\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\, \frac{\partial q_i}{\partial \alpha}\frac{d}{dt}\left(\frac{\partial f}{\partial \dot q_i}\right).$$

vanishes at end-points

However, if we do require both $\delta q_i = 0$ & $\delta p_i = 0$, we can add $\frac{d}{dt}\underbrace{F(q, p, t)}_{\text{arbitrary smooth f'n}}$ to f.

Indeed, $\int_{t_1}^{t_2} dt\, \frac{dF}{dt} = F\Big|_{t_1}^{t_2}$ and

$\delta \int_{t_1}^{t_2} dt\, \frac{dF}{dt} = 0$ since F does not vary at the ends as the paths are varied.

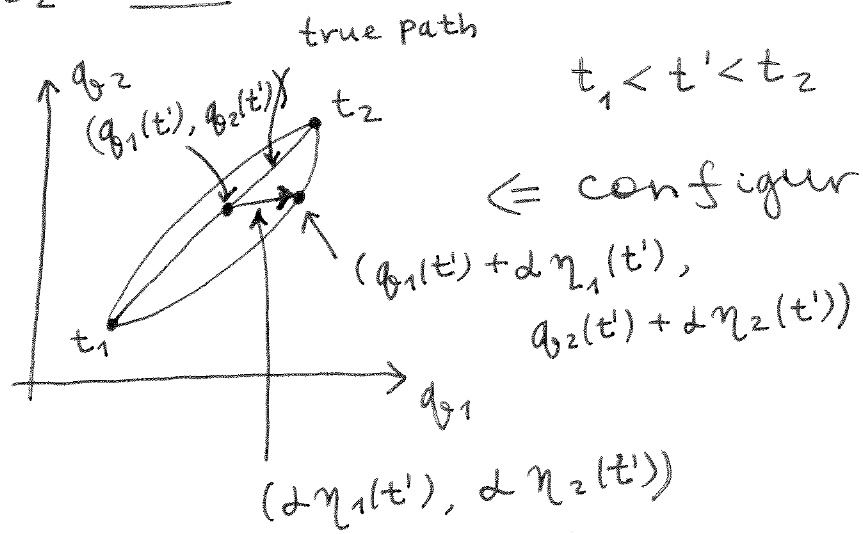
Then we can add $-\frac{d}{dt}(q_i p_i)$ to f, yielding

$$f = -\frac{d}{dt}(p_i q_i) + p_i \dot q_i - H = \underbrace{-\dot p_i q_i - H}_{\text{not } \mathcal{L} \text{ anymore}}$$

but will yield Hamilton's EoM

Principle of least action

Previously, we used δ-variation:
all paths started at t_1 & terminated
at t_2 and $\delta q_i(t_1) = \delta q_i(t_2) = 0$, $\forall i$.



$t_1 < t' < t_2$

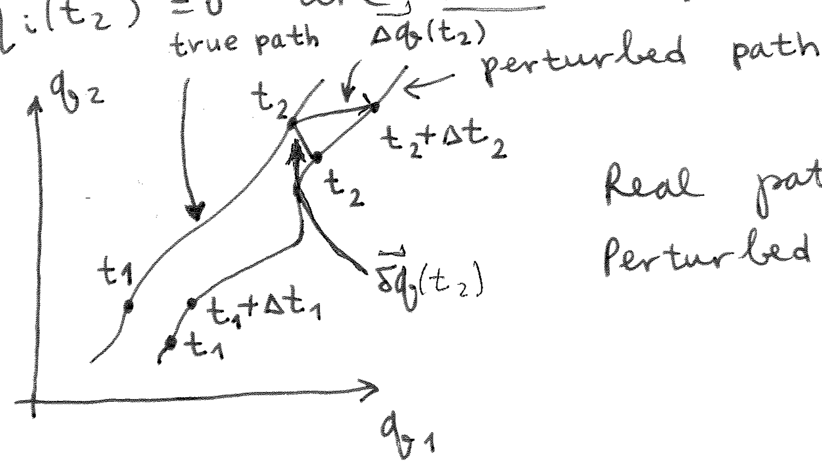
$\Leftarrow$ configuration space

Now consider Δ-variation:

$$q_i(t,\alpha) = \underbrace{q_i(t,0)}_{\text{true path}} + \alpha\eta_i(t)$$

$\eta_i(t)$ is smooth but $\eta_i(t_1) = 0$,

$\eta_i(t_2) = 0$ are not enforced:



Real path: (t_1, t_2)
Perturbed path: $(t_1 + \Delta t_1, t_2 + \Delta t_2)$

Now, consider

$$\Delta \int_{t_1}^{t_2} dt\, \mathcal{L} \equiv \underbrace{\int_{t_1+\Delta t_1}^{t_2+\Delta t_2} dt\, \mathcal{L}\Big|_\alpha}_{\text{to } 1^{\text{st}} \text{ order,}} - \int_{t_1}^{t_2} dt\, \underbrace{\mathcal{L}\Big|_{\alpha=0}}_{\text{"original" } \mathcal{L}} \; (=)$$

$\Delta t_1, \Delta t_2, \alpha$ are small parameters

$$\mathcal{L}(t_2)\Big|_{\alpha=0} \Delta t_2 - \mathcal{L}(t_1)\Big|_{\alpha=0} \Delta t_1 + \int_{t_1}^{t_2} dt\, \mathcal{L}\Big|_\alpha$$

$$(=) \; \mathcal{L}(t_2)\Delta t_2 - \mathcal{L}(t_1)\Delta t_1 + \int_{t_1}^{t_2} dt \Big[\underbrace{\mathcal{L}\Big|_\alpha - \mathcal{L}\Big|_{\alpha=0}}_{\text{"}\delta\mathcal{L}}\Big].$$

Now, $$\int_{t_1}^{t_2} dt\, \delta\mathcal{L} = \int_{t_1}^{t_2} dt \left[\frac{\partial \mathcal{L}}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha}\right] d\alpha =$$

$$= \int_{t_1}^{t_2} dt \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)}_{\text{"}0 \text{ due to EL EoM}}\Big] \underbrace{\frac{\partial q_i}{\partial \alpha} d\alpha}_{\delta q_i} +$$

$$+ \underbrace{\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{p_i} \underbrace{\frac{\partial q_i}{\partial \alpha} d\alpha}_{\delta q_i} \Big|_{t_1}^{t_2}}_{\text{no longer } =0 \text{ automatically}}.$$

So, $$\Delta \int_{t_1}^{t_2} dt\, \mathcal{L} = (\mathcal{L}\Delta t + p_i \delta q_i)\Big|_{t_1}^{t_2}$$

Next, we want to switch from δq_i's to Δq_i's:

$$\Delta q_i(t_2) \equiv q_i(t_2+\Delta t_2, \alpha) - q_i(t_2, 0) =$$

$$= q_i(t_2+\Delta t_2, 0) + \underbrace{\alpha\, \eta_i(t_2+\Delta t_2)}_{\substack{\approx \alpha \eta_i(t_2) \text{ to} \\ 1^{st} \text{ order in } (\alpha, \Delta t_2)}} - q_i(t_2, 0) \approx$$

$$\approx \dot{q}_i(t_2)\Delta t_2 + \delta q_i(t_2)$$

[can see it on the plot above as well]

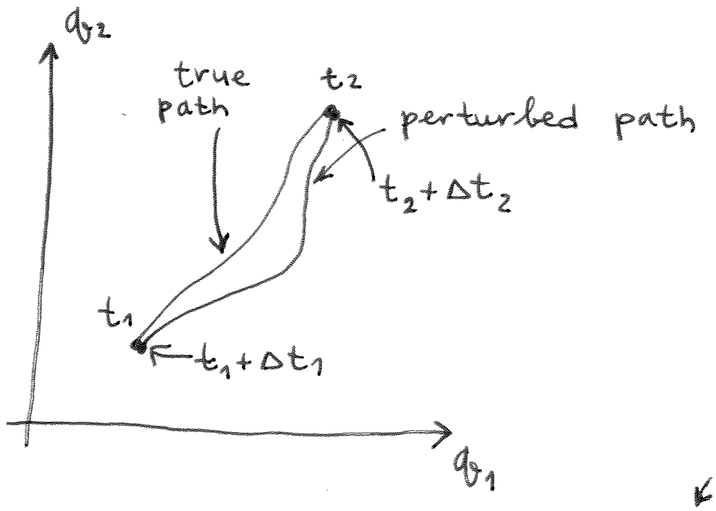
Hence
$$\Delta \int_{t_1}^{t_2} dt\, L = \left(L \Delta t + p_i \Delta q_i - p_i \dot{q}_i \Delta t\right)\Big|_{t_1}^{t_2} =$$

$$= \left(p_i \Delta q_i - H \Delta t\right)\Big|_{t_1}^{t_2}.$$

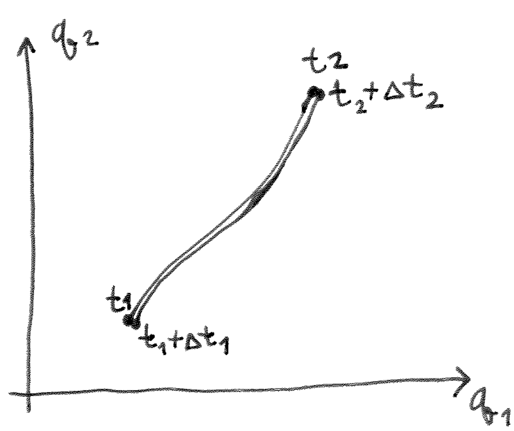
Now, focus on the following systems:

1. H is conserved [H & L are not explicit f's of t]
2. H is conserved on all perturbed paths (but does not have to be equal to H of the real path!)
3. Require that $\Delta q_i(t_2) = 0$, $\Delta q_i(t_1) = 0$ but Δt_1 & $\Delta t_2 \neq 0$ in general

graphically, we have:



For example, the two paths (in configuration space) may be exactly the same for both ~~pattern~~ systems:



If $H = \dfrac{p_1^2}{2m} + \dfrac{p_2^2}{2m} + V(q_1, q_2)$, "perturbed" H is say

$$H' = \frac{p_1'^2}{2m} + \frac{p_2'^2}{2m} + V(q_1, q_2).$$

If $p_1' > p_1$ & $p_2' > p_2$,

($H'>H$) $\dot{q}_1' > \dot{q}_1$ & $\dot{q}_2' > \dot{q}_2$.

Then the perturbed curve is traversed faster and, if both curves start at $t=0$, $\Delta t_1 < 0$ & $\Delta t_2 < 0$ (the perturbed curve arrives first at both the beginning and the end of the common trajectory).

With these conditions,

$$\Delta \int_{t_1}^{t_2} dt\, \mathcal{L} = -H(\Delta t_2 - \Delta t_1).$$

But $\displaystyle\int_{t_1}^{t_2} dt\, \mathcal{L} = \int_{t_1}^{t_2} dt\, p_i \dot{q}_i - \underbrace{H}_{\substack{\text{const}\\ \text{by construction}}}(t_2 - t_1)$ and

$$\Delta \int_{t_1}^{t_2} dt\, \mathcal{L} = \Delta \int_{t_1}^{t_2} dt\, p_i \dot{q}_i - H\Big|_{\alpha}(t_2 + \Delta t_2 - t_1 - \Delta t_1) +$$

$$+ H\Big|_{\alpha=0}(t_2 - t_1) = \Delta \int_{t_1}^{t_2} dt\, p_i \dot{q}_i - H\Big|_{\alpha=0}(\Delta t_2 - \Delta t_1) +$$

$$+ (t_2 - t_1)\left[H\Big|_{\alpha=0} - H\Big|_{\alpha} \right]$$

In the special case of $H\big|_{\alpha} = H$ (i.e. all paths have exactly the same energy)

$$\Delta \int_{t_1}^{t_2} dt\, \mathcal{L} = \Delta \int_{t_1}^{t_2} dt\, p_i \dot{q}_i - H(\Delta t_2 - \Delta t_1),$$

yielding

$$\left.\underline{\underline{\Delta \int_{t_1}^{t_2} dt\, p_i \dot{q}_i = 0}}\ . \quad (*)\ \right] \text{ principle of least action}$$

If $\mathcal{L} = T - V = \frac{1}{2} M_{jk}(q)\, \dot{q}_j \dot{q}_k - V(q)$,

$p_i = M_{ij}(q)\, \dot{q}_j$, so that

$p_i \dot{q}_i = 2T$ and $(*)$ becomes

$$\underline{\underline{\Delta \int_{t_1}^{t_2} dt\, T = 0}}\ .$$

If $T = \text{const}$ (no external forces), we have:

$$\underbrace{\Delta (t_2 - t_1) = 0}_{\text{real path has the least transit time}}$$

Consider $$T = \frac{1}{2} M_{jk}(q)\, \dot{q}_j \dot{q}_k$$

In analogy with $ds^2 = \underbrace{g_{\mu\nu}}_{\text{metric tensor}} dx^\mu dx^\nu$,

define $dp^2 = M_{jk}\, dq_j\, dq_k$, then

$$\left(\frac{dp}{dt}\right)^2 = M_{jk}\, \dot{q}_j \dot{q}_k \quad \text{and}$$

$$T = \frac{1}{2}\left(\frac{dp}{dt}\right)^2$$

$$\Downarrow$$

$$dt = \frac{dp}{\sqrt{2T}}$$

But then $$\Delta \int_{t_1}^{t_2} dt\, T = \Delta \int_{p_1}^{p_2} dp \sqrt{\frac{T}{2}} = 0 .$$

This leads to $$\underbrace{\Delta \int_{p_1}^{p_2} dp \sqrt{\underset{\substack{\Uparrow \\ \text{const}}}{H} - V(q)} = 0}_{\text{Jacobi's principle of least action}} .$$

Here, we have curvilinear configuration space characterized by a metric tensor M_{jk}. The system traces a path in this configuration space with speed $\frac{dp}{dt} = \sqrt{2T}$

If $T = \text{const}$ $\quad (V(q) = 0)$,

$\Delta(p_2 - p_1) = 0$ if the system travels along the shortest path in the configuration space (i.e. along the geodesics).

Equivalently, if $T = \text{const}$, the system travels along the path of least curvature.

$\Uparrow$

Hertz's principle of least curvature