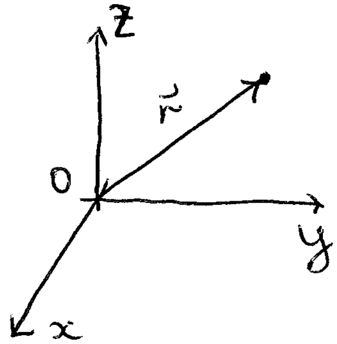
Lecture 1

Single-particle mechanics



$\vec{v} = \frac{d\vec{r}}{dt}$ particle velocity

$\vec{p} = m\vec{v}$ linear momentum

(m: part. mass)

EoM: (*) $\underbrace{\frac{d\vec{p}}{dt}}_{\dot{\vec{p}}} = \underbrace{\vec{F}}_{\text{total force acting on particle}}$ $\Leftarrow$ Newton's 2nd law

So, $\vec{F} = \frac{d}{dt}(m\vec{v}) \underset{m = \text{const}}{=} m\frac{d\vec{v}}{dt} = m\vec{a}$, where $\underbrace{\vec{a}}_{\text{part. acceleration}} = \frac{d^2\vec{r}}{dt^2}$

So, EoM is typically a 2nd order DE in $\vec{r}$.

(*) is valid in an inertial reference frame.

Conservation laws:

if $\vec{F} = 0$, $\vec{p} = \text{const}$

Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$

Torque: $\vec{N} = \vec{r} \times \vec{F}$

} defs

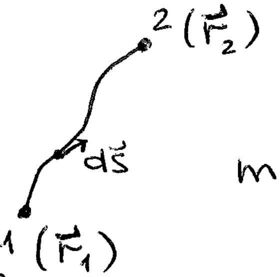
Consider $\vec{r} \times \vec{F} = \vec{r} \times \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d\vec{L}}{dt}$

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = \underbrace{\vec{v} \times m\vec{v}}_{=0} + \vec{r} \times \frac{d}{dt}(m\vec{v})$$

So, if $\vec{N} = 0$, $\vec{L} = \text{const}$

Next, consider

$$\underbrace{W_{12}}_{\text{work done on the particle (by def)}} = \int_1^2 d\vec{s} \cdot \vec{F}$$



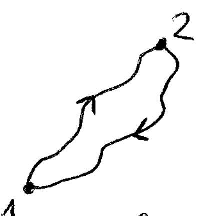
$m = \text{const}$

$$W_{12} = \int_1^2 d\vec{s} \cdot \vec{F} = m \int_1^2 \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt = \frac{m}{2} \int_1^2 \frac{d}{dt}(v^2)\, dt = \frac{m v_2^2}{2} - \frac{m v_1^2}{2} =$$

$= T_2 - T_1$, where $T = \frac{m v^2}{2}$ is the particle's kinetic energy

note that we used $\frac{d\vec{s}}{dt} = \vec{v}$ here, as well as $\vec{F}(\vec{r}) = m\vec{a}$ [no dissipation]

If, instead, we use a different path in the force field $\vec{F}(\vec{r})$, we again obtain $W_{12} = T_2 - T_1$ or, equivalently, $\oint \vec{F} \cdot d\vec{s} = 0$



But then $\vec{F}(\vec{r}) = -\vec{\nabla} V(\vec{r})$, where $V(\vec{r})$ is the potential energy.

Indeed, if W_{12} is indep. of the path, we must have $\vec{F} \cdot d\vec{s} = -dV$ (so that $\int_1^2 d\vec{s} \cdot \vec{F}$ depends only on the endpoints), and thus

$$F_s = -\frac{\partial V}{\partial s}.$$

Note that $V(\vec{r})$ is defined up to a const.

Finally, $W_{12} = V_1 - V_2 = T_2 - T_1$, or

$$\underbrace{T_1 + V_1}_{\substack{\text{total } E \\ \text{energy}}} = \underline{\underline{T_2 + V_2}}$$ energy conservation (forces must be conservative)

If $V = V(\vec{r}, t)$, E is no longer conserved in general.

Mechanics of a system of particles:

$$\sum_{j \neq i} \underbrace{\vec{F}_{ji}}_{\substack{\text{internal} \\ \text{force exerted} \\ \text{by } j\text{th particle}}} + \underbrace{\vec{F}_i^{(e)}}_{\text{external force}} = \dot{\vec{p}}_i$$] EoM for particle i

Then $$\underset{\substack{\uparrow \\ \text{sum over } i}}{\frac{d^2}{dt^2}} \sum_i m_i \vec{r}_i = \overbrace{\sum_i \vec{F}_i^{(e)}}^{\vec{F}^{(e)},\ \text{total external force}} + \underbrace{\sum_{\substack{i,j \\ j \neq i}} \vec{F}_{ji}}_{= 0}$$

$= 0$ since $\vec{F}_{ij} + \vec{F}_{ji} = 0$, $\forall i,j$ by assumption (Newton's 3rd law of motion)

Now, define $$\underset{\substack{\uparrow \\ \text{center} \\ \text{of mass}}}{\vec{R}} = \frac{\sum_i m_i \vec{r}_i}{\underbrace{\sum_i m_i}_{M,\ \text{total mass}}}$$

Then $$M \frac{d^2 \vec{R}}{dt^2} = \vec{F}^{(e)}$$] EoM for the center of mass (CoM)

Examples: exploding shell, rocket propulsion

Further, $\vec{P} = \sum_i m_i \frac{d\vec{r}_i}{dt} = M \frac{d\vec{R}}{dt}$.

$\uparrow$ total linear momentum

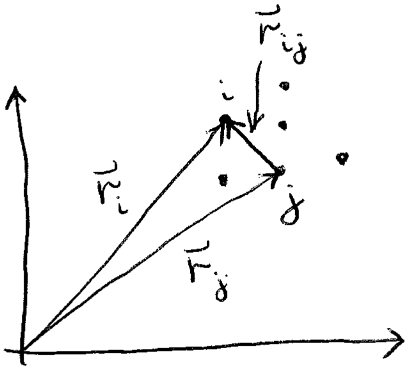
Thus, if $\vec{F}^{(e)} = 0$, $\vec{P} = \text{const}$

Now, consider $\sum_i \vec{r}_i \times \overset{\sum_{j\neq i} \vec{F}_{ji} + \vec{F}_i^{(e)}}{\dot{\vec{p}}_i} \underset{\vec{v}_i \times \vec{p}_i = 0,\ \forall i}{=} \sum_i \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) =$

$= \frac{d}{dt}\Big(\underbrace{\sum_i \vec{r}_i \times \vec{p}_i}_{\vec{L}\text{, total angular momentum}}\Big)$.

So, $\frac{d\vec{L}}{dt} = \underbrace{\sum_i \vec{r}_i \times \vec{F}_i^{(e)}}_{\vec{N}^{(e)}\text{ torque}} + \sum_{\substack{i,j\\ j\neq i}} \vec{r}_i \times \vec{F}_{ji}$

for each pair i,j: $\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = \underbrace{(\vec{r}_i - \vec{r}_j)}_{\vec{r}_{ij}} \times \vec{F}_{ji}$

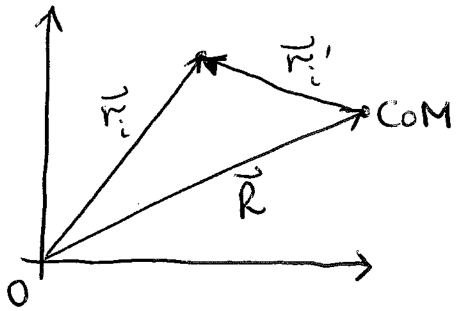


Often, $\vec{r}_{ij} \times \vec{F}_{ji} = 0$ (i.e., $\vec{r}_{ij} \uparrow\uparrow$ or $\uparrow\downarrow$ $\vec{F}_{ji}$)

Then $\frac{d\vec{L}}{dt} = \vec{N}^{(e)}$, and

if $\vec{N}^{(e)} = 0$, $\vec{L} = \text{const}$

Recall that $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$



Consider $\vec{r}_i = \vec{r}_i' + \vec{R}$,

$$\vec{v}_i = \underbrace{\vec{v}_i'}_{\frac{d\vec{r}_i'}{dt}} + \underbrace{\vec{v}}_{\frac{d\vec{R}}{dt}}$$

Then $\vec{L} = \sum_i \vec{R} \times (m_i \vec{v}) + \sum_i \vec{r}_i' \times (m_i \vec{v}) +$

$$+ \sum_i \vec{r}_i' \times (m_i \vec{v}_i') + \sum_i \vec{R} \times (m_i \vec{v}_i') =$$

$$= \vec{R} \times (M\vec{v}) + \sum_i \vec{r}_i' \times \vec{p}_i' + \underbrace{\Big(\sum_i m_i \vec{r}_i'\Big)}_{=0} \times \vec{v} \oplus$$

$$\oplus\ \vec{R} \times \frac{d}{dt} \underbrace{\Big(\sum_i m_i \vec{r}_i'\Big)}_{=0}.$$

So, $\vec{L} =$ CoM motion $+$ motion <u>about</u> CoM, in general depends on the origin O. However, if $\vec{v} = \frac{d\vec{R}}{dt} = 0$ (CoM at rest wrt O),

$$\vec{L} = \sum_i \vec{r}_i' \times \vec{p}_i'$$, angular momentum taken about the CoM

Finally, let's consider energy:

$$W_{12} = \sum_i \int_1^2 d\vec{s}_i \cdot \vec{F}_i = \sum_i \int_1^2 \vec{v}_i dt \cdot (m_i \dot{\vec{v}}_i) =$$

$$= \sum_i \int_1^2 dt \frac{d}{dt}\left(\frac{m_i v_i^2}{2}\right) = T_2 - T_1, \text{ where}$$

$$T = \frac{1}{2}\sum_i m_i \underbrace{v_i^2}_{=\vec{v}_i^{\,2}}$$ is the total kinetic energy.

Note that

$$T = \frac{1}{2}\sum_i m_i (\vec{v}+\vec{v}_i{}')\cdot(\vec{v}+\vec{v}_i) = \frac{1}{2}\sum_i m_i v^2 +$$

$$+ \frac{1}{2}\sum_i m_i v_i'^2 + \vec{v}\cdot \cancel{\frac{d}{dt}\Big(\underbrace{\sum_i m_i \vec{r}_i{}'}_{=0}\Big)}.$$

So, $$T = \underbrace{\frac{Mv^2}{2}}_{\text{CoM}} + \underbrace{\frac{1}{2}\sum_i m_i v_i'^2}_{\text{relative to CoM}}$$

Now, $$W_{12} = \sum_i \int_1^2 d\vec{s}_i \cdot \vec{F}_i = \sum_i \int_1^2 d\vec{s}_i \cdot \vec{F}_i^{(e)} +$$

$$+ \sum_{\substack{i,j\\ j\neq i}} \int_1^2 d\vec{s}_i \cdot \vec{F}_{ji} =$$

conservative forces:

$$= -\sum_i \int_1^2 d\vec{s}_i \cdot (\vec{\nabla}_i V_i) = \quad \begin{cases} \vec{F}_i^{(e)} = -\vec{\nabla}_i V_i, \\ V_{ij} = V_{ij}(|\vec{r}_i - \vec{r}_j|) = V_{ji} \end{cases}$$

$$\Downarrow$$

$$\underbrace{\vec{F}_{ji} = -\vec{\nabla}_i V_{ij} = \vec{\nabla}_j V_{ji} = -\vec{F}_{ij}}_{\text{internal forces equal \& opposite}}$$

Indeed,

forces $\uparrow\downarrow$ or $\uparrow\downarrow$ to $\vec{r}_i - \vec{r}_j$, equal & opposite

$$\begin{cases} \vec{\nabla}_i V_{ij}(|\vec{r}_i - \vec{r}_j|) = V_{ij}' \,\vec{\nabla}_i |\vec{r}_i - \vec{r}_j| = \\ \qquad = V_{ij}' \dfrac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|}, \\ \vec{\nabla}_j V_{ij}(|\vec{r}_i - \vec{r}_j|) = -V_{ij}' \dfrac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|} \end{cases}$$

to prevent overcounting of pairs

$$\stackrel{\text{to prevent}}{\frac{1}{2}} \sum_{\substack{i,j \\ j\neq i}} \int_1^2 \vec{\nabla}_{ij} V_{ij} \cdot d\vec{r}_{ij} = -\sum_i V_i \Big|_1^2 - \frac{1}{2} \sum_{\substack{i,j \\ j\neq i}} V_{ij} \Big|_1^2 .$$

for each pair of particles, we have

$$-\int_1^2 (\vec{\nabla}_i V_{ij} \cdot d\vec{s}_i + \vec{\nabla}_j \underbrace{V_{ij}}_{= V_{ji}} \cdot d\vec{s}_j) =$$

Recall that $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, yielding

$$\vec{\nabla}_i V_{ij} = -\vec{\nabla}_j V_{ij} = \underbrace{\vec{\nabla}_{ij}}_{\text{wrt } \vec{r}_i - \vec{r}_j} V_{ij}$$

$$= -\int_1^2 \vec{\nabla}_i V_{ij} \cdot \underbrace{(d\vec{s}_i - d\vec{s}_j)}_{d\vec{r}_{ij}}$$

The total potential energy is given by.

$$V = \sum_i V_i + \overbrace{\frac{1}{2} \sum_{\substack{i,j \\ j\neq i}} V_{ij}}^{\text{internal pot'l energy}}, \text{ and}$$

internal pot'l energy of the system, const in rigid bodies

$$T_2 - T_1 = -V_2 + V_1, \text{ or}$$

$$T_1 + V_1 = T_2 + V_2$$

total energy $T+V$ is conserved