Hamiltonian EoM {Lecture 19}

Recall Lagrangian EoM:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

n 2^{nd} order eq's, complemented by $2n$ initial conditions, e.g. n q_i's & n $\dot{q}_i$'s at $t=0$.

System evolves in n-dim configuration space defined by q_i's.

Hamiltonian formulation: seek 1^{st} order eq's of motion, $2n$ indep. variables.

System evolves in $2n$-dim phase space.

Typically, we choose n q_i's and n p_i's:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad j = 1, \dots, n$$

$(\{p\}, \{q\})$ are known as canonical variables.

We need to go from $(\{q\}, \{\dot{q}\}, t)$ to $(\{p\}, \{q\}, t) \Rightarrow$ use Legendre transform.

Consider $f(x,y)$:

$$df = u\,dx + v\,dy \quad , \text{ where}$$

$$u = \frac{\partial f}{\partial x} \quad , \quad v = \frac{\partial f}{\partial y}.$$

Suppose we need to change vars from (x,y) to (u,y). Consider

$$g = f - ux \quad , \text{ s.t.}$$

$$dg = \underbrace{df}_{u\,dx + v\,dy} - u\,dx - x\,du = v\,dy - x\,du \quad , \text{ or}$$

$$x = -\frac{\partial g}{\partial u} \quad , \quad v = \frac{\partial g}{\partial y}.$$

Clearly, $g = g(u,y)$ as desired.

Legendre transforms are used a lot in thermodynamics:

$$dU = dQ - dW \quad \Leftarrow 1^{\text{st}} \text{ law of thermodynamics}$$

For a gas,

$$dU = T\,dS - p\,dV \quad , \text{ so that } U = U(S,V).$$

Moreover, $T = \frac{\partial U}{\partial S}$ & $p = -\frac{\partial U}{\partial V}$.

We can transform from U to $H = U + pV$:

$$dH = \underbrace{dU}_{T\,dS - p\,dV} + p\,dV + V\,dp = T\,dS + V\,dp \quad , \text{ s.t.}$$

$$T = \frac{\partial H}{\partial S} \quad \& \quad V = \frac{\partial H}{\partial p}.$$

Clearly, $H = H(S, p)$.

Two more transforms are used:

$$\begin{cases} F = E - TS & \Leftarrow F(T, V) \quad \text{Helmholtz free en.} \\ G = H - TS & \Leftarrow G(T, p) \quad \text{Gibbs free en.} \end{cases}$$

Similarly, we write

$$d\mathcal{L} = \underbrace{\frac{\partial \mathcal{L}}{\partial q_i} dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i}_{\text{sum over } i} + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \Rightarrow \dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} \Leftarrow \text{Lagrange EoM}$$

Then $d\mathcal{L} = \dot{p}_i dq_i + p_i d\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t} dt$.

Legendre transform:

$$H(\{p\},\{q\},t) = p_i \dot{q}_i - \mathcal{L}(\{q\}, \{\dot{q}\}, t),$$ yielding

$$dH = \dot{q}_i dp_i + \cancel{p_i d\dot{q}_i} - \dot{p}_i dq_i - \cancel{p_i d\dot{q}_i} - \frac{\partial \mathcal{L}}{\partial t} dt =$$

$$= \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial \mathcal{L}}{\partial t} dt.$$

Since $dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$,

we obtain:

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}, \\ -\dfrac{\partial \mathcal{L}}{\partial t} = \dfrac{\partial H}{\partial t} \end{cases}$$

$2n+1$ eq's $\Rightarrow$

$\Rightarrow$ Hamilton (or canonical) EoM

Now recall the energy f'n:

$$h(\{q\}, \{\dot{q}\}, t) = \dot{q}_j \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_j}}_{p_j} - \mathcal{L} .$$

We've shown that $\dfrac{dh}{dt} = -\dfrac{\partial \mathcal{L}}{\partial t}$

Recall also that if

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$, where $\mathcal{L}_n$ is a homogeneous f'n in $\{\dot{q}\}$ of degree n,

$h = \mathcal{L}_2 - \mathcal{L}_0$.

If, further, $\mathcal{L}_2 = T$ & $\mathcal{L}_0 = -V$,

$$h = T + V = \underbrace{E}_{\text{total energy}}$$

So h is the same as H <u>except</u> $h = h(\{q\}, \{\dot{q}\}, t)$ and $H = H(\{p\}, \{q\}, t)$.

Constructing H:

1. Start with $\mathcal{L} = T - V$
2. Compute $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \equiv f_i(\{q\}, \{\dot{q}\}, t)$

3. Use $H = p_i \dot{q}_i - \mathcal{L}$ to compute H; it will be a f'n of both p_i's & $\dot{q}_i$'s
4. Invert the p_i eq's in step 2 to obtain $\dot{q}_j = \tilde{f}_j(\{p\}, \{q\}, t)$

5. Plug in the eq's from step 4 to obtain $H = H(\{p\}, \{q\}, t)$ (i.e. eliminate $\dot{q}_i$'s)

6. Use H from step 5 in the canonical EoM

Often (whenever $h = T + V$), we can use $H = T + V = \underbrace{E}_{\text{total en.}}$ directly but we still need to express H through p_i's rather than $\dot{q}_i$'s

We can make the procedure more explicit if

$$\mathcal{L} = \mathcal{L}_0(\{q\}, t) + a_i(\{q\}, t)\dot{q}_i + \underbrace{\tfrac{1}{2} T_i(\{q\}, t)\dot{q}_i^2}_{\substack{\text{no } q_i q_j,\ i \neq j \\ \text{terms} \\ \text{for simplicity}}}$$

Then $\mathcal{L} = \mathcal{L}_0 + \tilde{\dot{\vec{q}}}\,\vec{a} + \frac{1}{2}\tilde{\dot{\vec{q}}}\, T \,\dot{\vec{q}}$, where

$$\dot{\vec{q}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{pmatrix} \quad \text{and} \quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

T is a diagonal matrix.

For ex., if $\{q\} = (x, y, z)$ &

$$T = \begin{pmatrix} m & & 0 \\ & m & \\ 0 & & m \end{pmatrix}, \text{ we obtain:}$$

$$\frac{1}{2}\tilde{\dot{\vec{q}}}\, T\, \dot{\vec{q}} = \frac{1}{2} (\dot{x} \;\; \dot{y} \;\; \dot{z}) \begin{pmatrix} m & & 0 \\ & m & \\ 0 & & m \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} =$$

$$= \underline{\underline{\frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}}, \text{ as expected.}$$

Moreover,

$$\tilde{\dot{\vec{q}}}\,\vec{a} = (\dot{x} \;\; \dot{y} \;\; \dot{z}) \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \vec{a} \cdot \dot{\vec{r}}, \text{ where } \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Next, $H = \tilde{\dot{\vec{q}}}\,\vec{p} - \mathcal{L} = \tilde{\dot{\vec{q}}}(\vec{p} - \vec{a}) - \frac{1}{2}\tilde{\dot{\vec{q}}}\, T\, \dot{\vec{q}} - \mathcal{L}_0$, and

$$\vec{p} = \vec{a} + T\dot{\vec{q}} \implies \underline{\underline{\dot{\vec{q}} = T^{-1}(\vec{p} - \vec{a})}}.$$

$\vec{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$

Clearly, $\tilde{\dot{\vec{q}}} = (\tilde{\vec{p}} - \tilde{\vec{a}})\underbrace{T^{-1}}_{T^{-1} = \widetilde{T^{-1}}}$

Finally,

$$H = (\tilde{\vec{p}} - \tilde{\vec{a}}) T^{-1} (\vec{p} - \vec{a}) - \frac{1}{2} (\tilde{\vec{p}} - \tilde{\vec{a}}) \underbrace{T^{-1} T}_{\mathbb{I}} T^{-1} (\vec{p} - \vec{a}) -$$

$$- \mathcal{L}_0 = \frac{1}{2} (\tilde{\vec{p}} - \tilde{\vec{a}}) T^{-1} (\vec{p} - \vec{a}) - \mathcal{L}_0 . \qquad (*)$$

In our example above,

$$T^{-1} = \begin{pmatrix} m^{-1} & & 0 \\ 0 & m^{-1} & \\ & & m^{-1} \end{pmatrix} .$$

Ex. Consider $\mathcal{L} = \frac{m}{2} (\dot{r}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2 + r^2 \dot{\theta}^2) - V(r)$

$\mathcal{L} = T - V$

$\{r, \theta, \varphi\}$ are generalized coords

Clearly, $H = T + V$

$$\begin{cases} \mathcal{L}_0 = -V(r), \\ \vec{a} = \vec{0} \end{cases} \qquad T = \begin{pmatrix} m & & 0 \\ & mr^2 & \\ 0 & & mr^2 \sin^2\theta \end{pmatrix}$$

$$\Downarrow$$

$$T^{-1} = \begin{pmatrix} \frac{1}{m} & & 0 \\ & \frac{1}{mr^2} & \\ 0 & & \frac{1}{mr^2 \sin^2\theta} \end{pmatrix}$$

Acc. to eq. (*),

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta} \right) + V(r) . \qquad (**)$$

Alternatively, just use

$$\begin{cases} p_r = \dfrac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \dfrac{p_r}{m}, \\ p_\theta = \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \dfrac{p_\theta}{mr^2}, \\ p_\varphi = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mr^2\sin^2\theta\,\dot{\varphi} \Rightarrow \dot{\varphi} = \dfrac{p_\varphi}{mr^2\sin^2\theta}. \end{cases}$$

Then $T = \frac{m}{2}(\dot{r}^2 + r^2\sin^2\theta\,\dot{\varphi}^2 + r^2\dot{\theta}^2) =$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right), \text{ and}$$

(**) is recovered.

For a system with n DoF, construct

$$\vec{\eta} = \underbrace{\begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix}}_{2n}$$

Then construct

$$\frac{\partial H}{\partial \vec{\eta}} = \underbrace{\begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_n} \\ \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \end{pmatrix}}_{2n}$$

Finally, define

$$J = \begin{pmatrix} \underset{\text{zero matrix}}{0} & \underset{\text{unit matrix}}{\mathbb{1}} \\ -\mathbb{1} & 0 \end{pmatrix}$$

Note that $J\tilde{J} = \tilde{J}J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$

$|J| = 1$, $\tilde{J} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = -J = J^{-1}$, etc.

Then $\dot{\vec{\eta}} = J \dfrac{\partial H}{\partial \vec{\eta}}$. For example, for $n=2$:

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \frac{\partial H}{\partial q_2} \\ \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_1} \end{pmatrix}$$

, which encapsulates $2n$ Hamilton EoM

For cyclic coordinates,

$\dot{p}_j = \dfrac{\partial \mathcal{L}}{\partial q_j} = 0$ ($\mathcal{L}$ does not depend on q_j)

$\Longrightarrow p_j = \text{const}$

But $\dot{p}_j = -\dfrac{\partial H}{\partial q_j} \Rightarrow \dfrac{\partial H}{\partial q_j} = 0$, q_j does not appear in H as well.

Next, consider

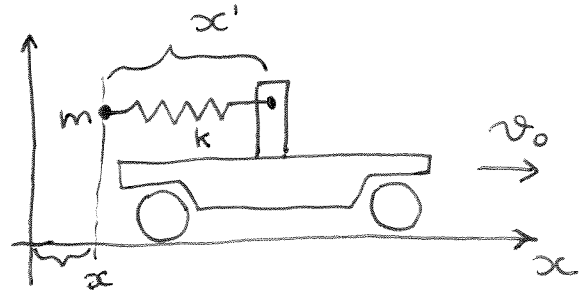
$$\frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial q_i}}_{-\dot{p}_i} \dot{q}_i + \underbrace{\frac{\partial H}{\partial p_i}}_{\dot{q}_i} \dot{p}_i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

But $H = p_i \dot{q}_i - \mathcal{L} \Rightarrow \underbrace{\frac{\partial H}{\partial t}}_{\text{(see above) Hamilton EoM}} = -\frac{\partial \mathcal{L}}{\partial t}$.

Therefore $\frac{\partial \mathcal{L}}{\partial t} = 0$ entails $\frac{\partial H}{\partial t} = 0$, yielding

$$\frac{dH}{dt} = 0 \Rightarrow H = \text{const}(t).$$

Ex.:



massless cart

Use x as a generalized coord: equil. spring length

$$\mathcal{L}(x, \dot{x}, t) = \frac{m\dot{x}^2}{2} - \frac{k}{2}(\underbrace{x - v_0 t}_{x'} - b)^2$$

EoM: $m\ddot{x} = -k(\underbrace{x - v_0 t - b}_{\text{"}x\text{"}})$

Using $\ddot{x} = \ddot{x}''$, we obtain:

$m\ddot{x}'' = -kx''$ simple harmonic motion in the cart frame

Now, $\underset{(p,x,t)}{H} = \frac{p^2}{2m} + \frac{k}{2}(x - v_0 t - b)^2$

$$\frac{\partial H}{\partial t} \neq 0 \Rightarrow H \neq \text{const}(t).$$

Total energy is <u>not</u> conserved: work must be done on the cart to keep it moving with constant speed v_0 while the mass is oscillating.

Now, consider $x'' = x - v_0 t - b$:

$$\dot{x} = \dot{x}'' + v_0 \quad , \text{ and}$$

$$\mathcal{L}(x'', \dot{x}'') = \frac{m\dot{x}''^2}{2} + m\dot{x}'' v_0 + \underbrace{\frac{m v_0^2}{2}}_{\text{const}} - \frac{k x''^2}{2} .$$

Then $$H'(p''; x'') \underset{\substack{\uparrow \\ \text{use } (*), \\ \text{p. 7}}}{=} \frac{1}{2m}(p'' - m v_0)^2 + \frac{k x''^2}{2} - \frac{m v_0^2}{2} . \qquad \left(\begin{smallmatrix} * \\ * \end{smallmatrix}\right)$$

$$\frac{\partial H'}{\partial t} = 0 \quad \Rightarrow \quad H' = \text{const}(t) \qquad (!)$$

$p'' - m v_0$ is momentum of mass m relative to the moving cart, so that the 1st 2 terms in $\left(\begin{smallmatrix} * \\ * \end{smallmatrix}\right)$ are the kinetic & potential energy in the cart's frame.

This energy is conserved as expected.

What about EoM?

$\underline{\underline{H}}$: $$\begin{cases} \dot{x} = \dfrac{p}{m} , \\ \dot{p} = -k(x - v_0 t - b) \end{cases} \Rightarrow m\ddot{x} = -k(x - v_0 t - b)$$

$\underline{\underline{H'}}$: $\begin{cases} \dot{x}'' = \dfrac{p'' - m v_0}{m}, \\ \dot{p}'' = -k x'' \end{cases}$ $\Rightarrow$ $p'' = m(\dot{x}'' + v_0)$,

$m\ddot{x}'' = -k x''$, or

$m\ddot{x} = -k(x - v_0 t - b)$,

exactly as above.