Lecture 16

Oscillations

Assume that the system is described by

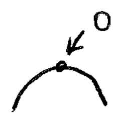
$\{q_1, \dots, q_n\} \Leftarrow$ general'd coords

At equil., $\quad Q_i = -\left(\frac{\partial V}{\partial q_i}\right)_0 = 0 \quad , \quad i = 1, \dots, n$

Equil. config'n: $\{q_{01}, q_{02}, \dots, q_{0n}\}$

stable if $V|_0$ is a min

unstable if $V|_0$ is a max



Focus on the motion of the system around a <u>stable</u> equil. configuration:

$$q_i = q_{0i} + \underbrace{\eta_i}_{\text{new generalized coords}}$$

$$V(q_1, \dots, q_n) = \underbrace{V(q_{01}, \dots, q_{0n})}_{\text{const}} + \cancel{\left(\frac{\partial V}{\partial q_i}\right)_0} \eta_i + \quad \text{at equil.}$$

$$+ \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j + \dots$$

if $V(q_1, \dots, q_n) \to V(q_1, \dots, q_n) - V(q_{01}, \dots, q_{0n})$,
i.e. shifted by a const

we obtain:

$$V = \frac{1}{2} \underbrace{\left(\frac{\partial^2 V}{\partial q_i \partial q_j} \right)_0}_{\equiv V_{ij}} \eta_i \eta_j = \frac{1}{2} V_{ij} \eta_i \eta_j$$

(+ higher-order terms)

Clearly, $V_{ij} = V_{ji}$

For kinetic energy, in general

$$T = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j = \underbrace{\frac{1}{2} m_{ij} \dot{\eta}_i \dot{\eta}_j}_{2^{\text{nd}} \text{ order in } \dot{\eta}_i\text{'s}}$$

Expand m_{ij}:

$$m_{ij}(q_1, \ldots, q_n) \simeq \underbrace{m_{ij}(q_{01}, \ldots, q_{0n})}_{\text{const}} + \left(\frac{\partial m_{ij}}{\partial q_k} \right)_0 \eta_k + \ldots$$

$$m_{ij}(q_{01}, \ldots, q_{0n}) \equiv T_{ij}$$

Thus $T = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j$ (+ higher-order terms)

Finally, $\mathcal{L} = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j - \frac{1}{2} V_{ij} \eta_i \eta_j$,

yielding

$$(*) \quad T_{ij} \ddot{\eta}_j + V_{ij} \eta_j = 0 \quad \Leftarrow \text{EoM} \quad i = 1, \ldots, n$$

no cross-terms in T

Often, $\mathcal{L} = \frac{1}{2} T_i \dot{\eta}_i^2 - \frac{1}{2} V_{ij} \eta_i \eta_j$, yielding

$$(**) \quad \underbrace{T_i \ddot{\eta}_i}_{\text{no sum over } i} + V_{ij} \eta_j = 0. \quad i = 1, \ldots, n$$

EoM (*) can be solved by:

$$\eta_i = \underbrace{a_i}_{\text{complex amplitude}} e^{i\omega t} \qquad \mathrm{Re}\{\eta_i\} \text{ gives physical motion}$$

$$V_{ij}a_j - \omega^2 T_{ij} a_j = 0 \quad \Leftarrow \text{ n homog. linear eq's for } a_j\text{'s}$$

Non-trivial solution requires

$$\begin{vmatrix} V_{11}-\omega^2 T_{11} & V_{12}-\omega^2 T_{12} & \cdots \\ V_{21}-\omega^2 T_{21} & \cdots & \\ \vdots & & \end{vmatrix} = 0$$

$\Uparrow$

can be used to obtain the ω's

In matrix form,

$$V\vec{a} = \underbrace{\omega^2}_{\equiv\lambda} T\vec{a} \quad \Leftarrow \text{"eigenvalue" eq'n}$$

$$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

adjoint (transpose + c.c.)

Then $\vec{a}_i^{+}* \Big| \; V\underbrace{\vec{a}_k}_{k\text{th eigenvector}} = \underbrace{\omega_k^2}_{\equiv\lambda_k} T\vec{a}_k$ and $\vec{a}_i^{+} V = \lambda_i^* \vec{a}_i^{+} T \; \Big| * \vec{a}_k$

$\Uparrow$

$V = V^{+}, \; T = T^{+}$

Subtract the eq's above (after multiplying as shown):

$$\lambda_k \vec{a}_i^{+} T \vec{a}_k - \lambda_i^* \vec{a}_i^{+} T \vec{a}_k =$$
$$= (\lambda_k - \lambda_i^*)\, \vec{a}_i^{+} T \vec{a}_k = \underline{\underline{0}}.$$

Choose $i=k$:

$$(\lambda_k - \lambda_k^*)\, \vec{a}_k^{\,+}\, T\, \vec{a}_k = 0 \qquad (\ast\ast)$$

Note that $(\vec{a}_k^{\,+} T \vec{a}_k)^+ = \vec{a}_k^{\,+} \underbrace{T^+}_{=T} \vec{a}_k$, so that $\vec{a}_k^{\,+} T \vec{a}_k$ is real.

Moreover, consider $\vec{a}_k = \vec{\alpha}_k + i\vec{\beta}_k$:

$$\vec{a}_k^{\,+} T \vec{a}_k = \tilde{\vec{\alpha}}_k T \vec{\alpha}_k + \tilde{\vec{\beta}}_k T \vec{\beta}_k +$$

$$+ i\underbrace{(\tilde{\vec{\alpha}}_k T \vec{\beta}_k - \tilde{\vec{\beta}}_k T \vec{\alpha}_k)}_{\alpha_{k,i} T_{ij} \beta_{k,j} - \underbrace{\beta_{k,i} \overbrace{T_{ij}}^{=T_{ji}} \alpha_{k,j}}_{\alpha_{k,j} T_{ji} \beta_{k,i}} = 0}, \text{ real as expected}$$

Both $\tilde{\vec{\alpha}}_k T \vec{\alpha}_k$ & $\tilde{\vec{\beta}}_k T \vec{\beta}_k$ are like kinetic energies for some generalized velocities $\vec{\alpha}_k$ & $\vec{\beta}_k$ $\Rightarrow$ these terms must be > 0 for non-zero velocities.

Then $\vec{a}_k^{\,+} T \vec{a}_k > 0 \Rightarrow \lambda_k = \lambda_k^*$ in $(\ast\ast)$,
i.e. all the eigenvalues are real

Next, consider

$$\underbrace{\lambda_k}_{\text{real}} = \frac{\overbrace{\vec{a}_k^{\,+} V \vec{a}_k}^{\text{pos. semi-def. (see below)}}}{\underbrace{\vec{a}_k^{\,+} T \vec{a}_k}_{\text{pos. def.}}} \Rightarrow \overset{=\omega_k^2}{\lambda_k} \geq 0 \text{ \& } \omega_k\text{'s are real}$$

Note that

$$\vec{a}_k^{\,+} V \vec{a}_k = \overbrace{\tilde{\vec{\alpha}}_k V \vec{\alpha}_k + \tilde{\vec{\beta}}_k V \vec{\beta}_k}^{\text{pos. semidefinite since } V \geq 0}, \text{ just as with } T$$

However, it's easy to see that $\vec{\alpha}_k$ & $\vec{\beta}_k$ have to be eigenvectors separately if λ_k is real. Indeed, consider

$$\underbrace{(V - \lambda_k T)}_{\text{real}} \cdot \underbrace{\vec{a}_k}_{\vec{\alpha}_k + i\vec{\beta}_k} = 0 \implies \begin{cases} (V - \lambda_k T) \cdot \vec{\alpha}_k = 0, \\ (V - \lambda_k T) \cdot \vec{\beta}_k = 0. \end{cases}$$

$\Downarrow$

$\vec{\alpha}_k$ & $\vec{\beta}_k$ are real eigenvectors, and $\vec{a}_k$ is simply a linear combination with complex coeff's

In the non-degenerate case (a single eigenvector for each λ_k), it therefore suffices to consider real eigenvectors: $\vec{a}_k \to \vec{\alpha}_k$. (i.e., $\vec{\beta}_k = \vec{0}$)

Even in the degenerate case, all the eigenvectors will be real. (but you can make complex linear combinations out of them)

Now, eq'n (**) just before gives:

$$(\lambda_k - \lambda_\ell) \tilde{\vec{a}}_\ell T \vec{a}_k = 0.$$

we focus on the non-degenerate case where $\lambda_k \neq \lambda_\ell$ if $k \neq \ell$

For $k \neq \ell$, $\tilde{\vec{a}}_\ell T \vec{a}_k = 0$.

We can also choose $\tilde{\vec{a}}_k T \vec{a}_k = 1$
(the scale is not set by the eigenvalue eq'n)

Thus $\tilde{A} T A = \mathbb{1}$, where each column of A is $\vec{a}_k$

Next, recall similarity transform:

$$C' = BCB^{-1} \qquad (+)$$

Analogously, introduce congruence transform:

$$C' = \tilde{A}CA \qquad (++)$$

If A is orthogonal, $\tilde{A} = A^{-1}$ and (++) becomes (+) if we choose $A^{-1} = B$.
[but in general (+) & (++) are different]

So, $\tilde{A}TA = \mathbb{1}$ is a congruence transform of T into $\mathbb{1}$.

$\lambda_{ik} = \lambda_k \delta_{ik}$

Introduce $\lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, then

$$V\vec{a}_k = \lambda_k T\vec{a}_k \Rightarrow V_{ij}a_{jk} = \underbrace{T_{ij}}_{=}\underbrace{a_{jk}\lambda_k}_{a_{jl}\underbrace{\lambda_l\delta_{lk}}_{\lambda_{lk}}}$$

δ_{ij} in reduced form (i.e., after applying $\tilde{A}TA = \mathbb{1}$)

In matrix notation,

$$\underbrace{VA}_{\text{each column is } \vec{a}_k} = T A\lambda, \text{ or}$$

$T = \mathbb{1}$ ← if $T = \mathbb{1}$

$\boxed{VA = \lambda A}$

$$\tilde{A}VA = \underbrace{\tilde{A}TA}_{\mathbb{1}}\lambda \Rightarrow \tilde{A}VA = \lambda$$

$$\Downarrow$$

$$VA = \underbrace{\tilde{A}^{-1}}_{TA}\lambda$$

otherwise $|V - \lambda T| = 0$

$$T = \mathbb{1} \Rightarrow |V - \lambda\mathbb{1}| = 0$$

number, not a matrix

As an example, consider

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2} V_{ij}\, x_i x_j .$$

$i,j = 1,2$

Introduce
$$\begin{cases} x_1' = x_1 \sqrt{m}, \\ x_2' = x_2 \sqrt{m} \end{cases}$$

Then $\Leftarrow$ reduced form

$$\mathcal{L} = \frac{1}{2}\left(\dot{x}_1'^2 + \dot{x}_2'^2\right) - \frac{1}{2} \underbrace{\frac{V_{ij}}{m}}_{\equiv V_{ij}'} x_i' x_j' .$$

Now, consider (dropping all 's)

$$\begin{vmatrix} V_{11}-\lambda & V_{12} \\ V_{21} & V_{22}-\lambda \end{vmatrix} = 0 \quad , \text{or}$$

$$(V_{11}-\lambda)(V_{22}-\lambda) - V_{12}V_{21} = 0 ,$$

$$\lambda^2 - (V_{11}+V_{22})\lambda + V_{11}V_{22} - V_{12}V_{21} = 0 , \quad \text{yielding}$$

$$\lambda_{1,2} = \frac{1}{2}\left((V_{11}+V_{22}) \pm \sqrt{(V_{11}-V_{22})^2 + 4V_{12}V_{21}} \right).$$

For simplicity, consider $V_{11}>0, V_{22}>0$ and $V_{11} > V_{22}$

$$0 \neq V_{21} = V_{12} \ll \underbrace{V_{11}-V_{22}}_{>0} .$$

Define
$$\underset{>0}{\delta} = \frac{V_{12}}{V_{11}-V_{22}} \ll 1, \quad \text{then}$$

$$\lambda_{1,2} = \frac{1}{2}\left((V_{11}+V_{22}) \pm (V_{11}-V_{22}) \sqrt{1 + \frac{4V_{12}^2}{(V_{11}-V_{22})^2}} \right) \approx$$

$$\approx \frac{V_{11}+V_{22}}{2} \pm \frac{V_{11} \mp V_{22}}{2}\left(1+\frac{2V_{12}^2}{(V_{11} \pm V_{22})^2}\right) =$$

$$= \frac{V_{11}+V_{22}}{2} \pm \frac{V_{11} \mp V_{22}}{2} \pm V_{12}\,\delta \,.$$

So, $\begin{cases} \lambda_1 = V_{11} + V_{12}\,\delta, & > 0 \\ \lambda_2 = V_{22} - V_{12}\,\delta. & > 0 \end{cases}$ as expected

The eigenvectors can also be found:

labels eigenvector

use $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ s.t.

$$a_{11}^2 + a_{21}^2 = 1, \quad a_{12}^2 + a_{22}^2 = 1$$

and employ

$\begin{cases} V_{11}a_{11} + V_{21}a_{21} = \lambda_1 a_{11}, \\ V_{21}a_{11} + V_{22}a_{21} = \lambda_1 a_{21} \end{cases}$ & similar for λ_2 and $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$