Lecture 13

Principal axis transformation

It's possible to transform the inertia tensor into the diagonal form:

$$I_D = \begin{pmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{pmatrix}$$

In this case,

$$\vec{L} = I_D \vec{\omega} = \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix}, \text{ and}$$

$$T = \frac{\vec{\omega} \cdot I \cdot \vec{\omega}}{2} = \frac{1}{2} \left[I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \right].$$

Explicitly, $\quad I_D = \underbrace{R I \tilde{R}}_{\text{similarity transform}}$

$R : xyz \rightarrow \underbrace{x'y'z'}_{\text{principal axes (or eigenvectors)}}$
of the inertia tensor:
$\vec{v}_1 \parallel x', \ \vec{v}_2 \parallel y', \ \vec{v}_3 \parallel z'$.

Indeed, $\quad I \vec{v}_i = \underbrace{\overset{I_i \text{ in } I_D}{\overset{\parallel}{\lambda_i}} \vec{v}_i}_{\text{no sum implied}}, \quad i = 1, 2, 3$

Then $|I - \lambda \mathbb{I}| = 0$, or, more explicitly,

$$\underbrace{\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{xy} & I_{yy}-\lambda & I_{zy} \\ I_{xz} & I_{zy} & I_{zz}-\lambda \end{vmatrix}}_{\text{cubic eq'n}} = 0 \quad \text{will yield} \quad \underbrace{I_1, I_2, I_3}_{\substack{\text{some or all 3} \\ \text{may be equal depending} \\ \text{on the symm. of the body}}}$$

Note also that in principal axes,

$$I_1 = I_{xx} = \sum_i m_i (y_i'^2 + z_i'^2) > 0 .$$

Similarly, $\underset{I_2}{\underset{\parallel}{I_{yy}}} > 0$; $I_3 = I_{zz} > 0$.

We also have to have

$$I_{xy} = -\sum_i m_i x_i' y_i' = 0$$ & so on for 2 other off-diagonal elements. This implies that principal axes are high-symmetry axes $\Rightarrow$ e.g., for each particle at (x_i', y_i') there must be a particle at $(x_i', -y_i')$ so that their sum vanishes (the particle masses must be equal as well).

Finally, recall that

$$\underbrace{\tilde{I}}_{\substack{\text{moment of inertia} \\ \text{around } \vec{n}}} = \vec{n} \cdot I \cdot \vec{n}$$

$\vec{n} = \alpha \vec{i} + \beta \vec{j} + \gamma \vec{k}$, where α, β, γ are direction cosines

Then $\tilde{I} = I_{11} \alpha^2 + I_{22} \beta^2 + I_{33} \gamma^2 +$

$+ 2 I_{12} \alpha \beta + 2 I_{23} \beta \gamma + 2 I_{13} \alpha \gamma$.

Define $\underset{\text{"}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}}{\vec{p}} = \frac{\vec{n}}{\sqrt{\tilde{I}}}$, then

$$\underbrace{\begin{aligned} &I_{11} p_1^2 + I_{22} p_2^2 + I_{33} p_3^2 + 2 I_{12} p_1 p_2 + \\ &+ 2 I_{23} p_2 p_3 + 2 I_{13} p_1 p_3 = 1 \end{aligned}}_{\text{eq'n of inertial ellipsoid in 3D p-space}}$$

There is an axis transform s.t. we have:

$I_1 p_1'^2 + I_2 p_2'^2 + I_3 p_3'^2 = 1$. $\Leftarrow$ normal form of the ellipsoid

This is the principal axis transform'n; I_1, I_2, I_3 are the lengths of (semi) axes of the inertial ellipsoid in normal form.

For ex., if $I_1 = I_2 = I_3$, the inertial ellipsoid becomes a sphere.

Euler equations of motion

Often, CoM is taken as the origin of the body frame (if the motion has a fixed point, then that fixed point can be taken instead).

In any event,

$$T = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2$$

If V can be similarly split, Lagrangian methods become convenient.

Alternatively, for rot'n about a fixed point or CoM, the Newtonian approach can be used:

$$\underbrace{\left(\frac{d\vec{L}}{dt}\right)_{lab}}_{\text{inertial frame}} = \vec{N} \quad \text{, or}$$

$$\left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} = \vec{N}$$

$\Downarrow$ drop "body" for simplicity

$$\frac{dL_i}{dt} + \epsilon_{ijk} \omega_j L_k = N_i$$

Using the principal axes, we have:

$$L_i = \underbrace{I_i \omega_i}_{\text{no sum!}}$$

Finally,

$$I_i \frac{d\omega_i}{dt} + e_{ijk}\,\omega_j \omega_k I_k = N_i \quad \text{or, equivalently,}$$

$$\begin{cases} I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1, \\ I_2 \dot{\omega}_2 - \omega_1 \omega_3 (I_3 - I_1) = N_2, \\ I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3 \end{cases}$$

Euler's EoM (for a rigid body with a fixed point)

If $I_1 = I_2 \Rightarrow I_3 \dot{\omega}_3 = N_3$.

If $N_3 = 0 \Rightarrow \omega_3 = \text{const}$

Then the first 2 eq's decouple from the 3rd.