Lecture 10

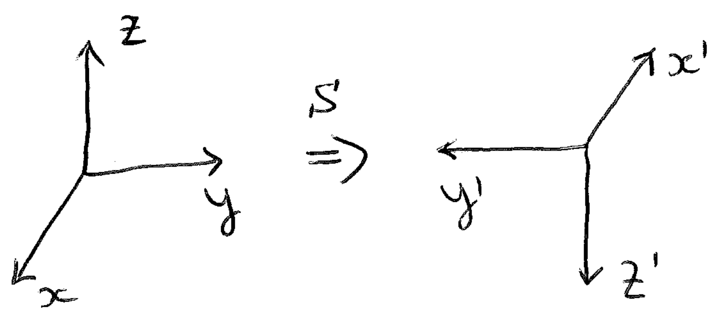
The Euler angles

Recall that $|A| = \pm 1$. However, only $|A| = 1$ corresponds to a rigid body rotation because A must evolve continuously from $\mathbb{I}$ & $|\mathbb{I}| = 1$. In fact, matrices with $|A| = -1$ correspond to an <u>inversion</u> of coord. axes.

For example, consider

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \Leftarrow |S| = -1$$

S transforms right-handed coords into left-handed:



Indeed,

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rot'n by } \pi \text{ around } z} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{\text{inversion of } z \text{ axis (reflection in the } xy \text{ plane)}}$$

does not correspond to any rotation

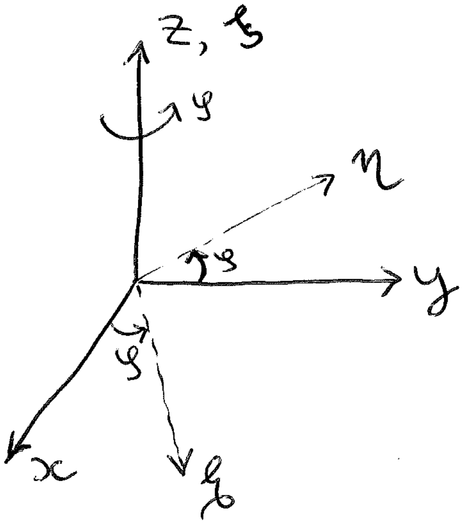
Orthogonal transforms with $|A|=1$ are called <u>proper</u>, $|A|=-1$ <u>improper</u>.

So, we need 3 indep. generalized coords $\Rightarrow$

$\Rightarrow$ Euler angles are commonly used: 3 successive rotations in a specific sequence.

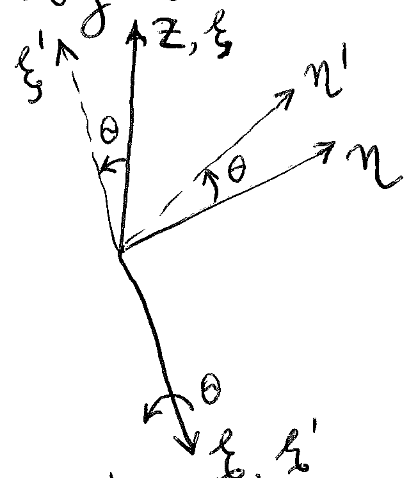
<u>One common convention:</u>

1. Rotate xyz by φ counter-clockwise around z:



$xyz \Rightarrow \xi\eta\zeta$

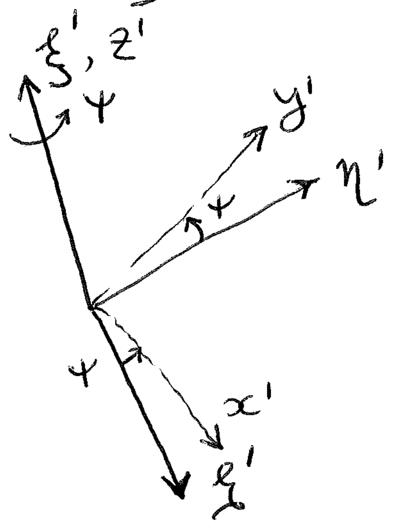
2. Rotate $\xi\eta\zeta$ by θ counter-clockwise around ξ:



$\xi\eta\zeta \Rightarrow \xi'\eta'\zeta'$

The ξ, ξ' axis is called the <u>line of nodes</u>

③ Rotate $\xi'\eta'\zeta'$ by ψ counter-clockwise around ζ':



$\xi'\eta'\zeta' \Rightarrow x'y'z'$

So, $\quad xyz \Rightarrow x'y'z'$
$\underbrace{(\varphi, \theta, \psi)}_{\text{Euler angles}}$

In matrix form,

$$\vec{\xi} = D\vec{x}, \quad \text{where } D = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{\xi} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{\xi}' = C\vec{\xi}, \quad \text{where } C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\vec{\xi}' = \begin{pmatrix} \xi' \\ \eta' \\ \zeta' \end{pmatrix}$$

$$\vec{x}' = B\vec{\xi}', \quad \text{where } B = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{x}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Finally, $\vec{x}' = A\vec{x}$, where

$$A = BCD = \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix}.$$

Further,

$$\vec{x} = A^{-1}\vec{x}' = \underbrace{\tilde{A}}_{\text{transpose of } A}\vec{x}'$$

The prescription above is a "z-x-z" prescription; sometimes, a "z-y-z" prescription is used. In engineering, "x-y-z" is often used, and the angles are called heading, pitch, roll. Note that 2 consecutive rotations cannot be around the same axis.

Euler's theorem on rigid body motion

With t, the orientation of the body will change: $A = \underbrace{A(t)}_{\text{continuous f'n of time, evolves from } \mathbb{I}}$ in general.

Note that $A(0) = \mathbb{I}$.

<u>Theorem</u>: General displacement of a rigid body with one point fixed is a rotation about some axis.

which goes through the fixed point

2 polar angles to describe the axis + 1 more angle to describe the rotation
(note: these are <u>not</u> Euler angles).

Rot'n around the axis leaves all vectors collinear with the axis unaffected:

$$\exists \vec{R} \text{ s.t. } \vec{R}' = A\vec{R} = \vec{R}$$

This is the $\lambda = 1$ case of the eigenvalue eq'n: $A\vec{R} = \lambda\vec{R} \Rightarrow (A - \lambda\mathbb{I})\vec{R} = 0$,

$$\vec{R} \neq 0 \quad \text{iff} \quad \underbrace{|A - \lambda\mathbb{I}| = 0}$$

characteristic eq'n, can be used to find $\lambda_1, \lambda_2, \lambda_3$

Then Euler's theorem reduces to the statement that one of the 3 λ's $\underline{\underline{=1}}$.

Define $\vec{R}_k = \begin{pmatrix} X_{1k} \\ X_{2k} \\ X_{3k} \end{pmatrix}$ $k=1,2,3$

eigenvector corresponding to λ_k

For $\vec{R}_k$, $A\vec{R} = \lambda\vec{R}$ gives

$$\sum_j a_{ij} X_{jk} = \lambda_k X_{ik} = \sum_j X_{ij} \delta_{jk} \lambda_k .$$

In matrix form,

$$AX = X\lambda \quad \text{, where}$$

$$\lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} \qquad X = \underbrace{\begin{pmatrix} \vec{R}_1 & \vec{R}_2 & \vec{R}_3 \end{pmatrix}}_{\text{each eigenvector is a column}}$$

Next, $\underbrace{X^{-1}AX}_{\text{similarity transform with } Y = X^{-1}} = \lambda$

Now, consider

$$(A - \mathbb{1})\tilde{A} = \mathbb{1} - \tilde{A} .$$

Then $|A - \mathbb{1}|\underbrace{|\tilde{A}|}_{+1 \text{ (proper rot'n)}} = |\mathbb{1} - \tilde{A}|$, and

$$|A-\mathbb{I}| = |\mathbb{I}-A|$$

$\nwarrow\ |\mathbb{I}-\tilde{A}| = |\tilde{\mathbb{I}}-\tilde{A}| = |\widetilde{\mathbb{I}-A}| = |\mathbb{I}-A|$

For any 3×3 matrix,

$$|-B| = (-1)^3 |B| = -|B|, \text{ so that}$$

$$\boxed{|A-\mathbb{I}| = 0} \quad (*)$$

So, we must have $|A-\lambda\mathbb{I}|=0$ for $\lambda=1$ $\nearrow$ $\exists$ nontrivial $\vec{R}$ which is left inv under a rotation

[b/c $(*)$ must hold for any rotation matrix A]. Note: does not hold in 2D space since $(-1)^2=1$, all vectors in the plane (2D) rotate

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Next, $X^{-1}AX = \lambda$ yields

$$\underbrace{|A|}_{=1 \text{ for a proper rot'n}} = |\lambda| = \lambda_1\lambda_2\underbrace{\lambda_3}_{=+1 \text{ say}} = 1, \text{ or}$$

$$\underline{\underline{\lambda_1\lambda_2 = 1}}.$$

$\begin{cases}\lambda_1 = a, \\ \lambda_2 = a^{-1}\end{cases}$ $a>1$ is excluded b/c it's a rotation

Since $\underbrace{A \text{ is real}}_{\text{\& hence } |A| \text{ is real}}$, we have $\begin{cases}\lambda_1 = \lambda \\ \lambda_2 = \lambda^*\end{cases}$

& $\underbrace{|\lambda|^2}_{=\lambda\lambda^*} = 1$

<u>3 cases:</u>

(1) $\lambda_1 = \lambda_2 = \lambda_3 = 1$, no rotation

(2) $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = 1$, rotation through π around z-axis

(3) $\lambda_1 = e^{i\phi}$, $\lambda_2 = e^{-i\phi}$, $\lambda_3 = 1$

Then $$\underbrace{\mathrm{Tr}(\lambda)}_{= 1+2\cos\varphi} = \mathrm{Tr}(X^{-1}AX) = \mathrm{Tr}(A)$$

If A indicates rotation around z-axis,

i.e. we choose the coord. system s.t. "some axis" in Euler's theorem is z-axis

$$A = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\mathrm{Tr}(A) = 1 + 2\cos\varphi$.

Thus φ is the rotation angle; case (1) is $\varphi = 0$, case (2) is $\varphi = \pi$.

Finally, note that if $\vec{R}$ is an eigenvector, so is $\underbrace{\alpha}_{\text{some const} \neq 0}\vec{R}$ & in particular $-\vec{R}$. Thus the sense of direction of Euler's rot'n axis is not specified. Moreover, $\varphi \to -\varphi$ does not change anything. So, some consistent conventions need to be employed.