1. $\boxed{5.4}$

Derive Euler's EoM:

$$\begin{cases} I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1, \\ I_2 \dot{\omega}_2 - \omega_1 \omega_3 (I_3 - I_1) = N_2, \\ I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3 \end{cases}$$

from the Lagrangian.

Lagrangian EoM:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j,$$ where $j = 1, \ldots, n$ labels generalized coords

Here, the generalized coords are (φ, ψ, θ).

Recall that

$$\begin{cases} \omega_1 = \dot{\varphi} \sin\theta \sin\psi + \dot{\theta} \cos\psi, \\ \omega_2 = \dot{\varphi} \sin\theta \cos\psi - \dot{\theta} \sin\psi, \\ \omega_3 = \dot{\varphi} \cos\theta + \dot{\psi}. \end{cases}$$ $\Leftarrow$ in the body frame

Moreover, $T = \underbrace{\frac{I_i \omega_i^2}{2}}_{\text{sum over } i=1,2,3 \text{ implied}}$

Start with the $\overset{\text{Lagrangian}}{\text{eq'n}}$ for ψ:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = \underbrace{N_\psi}_{\text{torque}}$$

ψ rotation is around z body axis, so we should obtain an eq'n for $\dot{\omega}_3$

Using
$$\begin{cases} \dfrac{\partial \omega_1}{\partial \psi} = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \\ \dfrac{\partial \omega_2}{\partial \psi} = -\dot{\varphi}\sin\theta\sin\psi - \dot{\theta}\cos\psi, \\ \dfrac{\partial \omega_3}{\partial \psi} = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial \omega_1}{\partial \dot{\psi}} = \dfrac{\partial \omega_2}{\partial \dot{\psi}} = 0, \\ \dfrac{\partial \omega_3}{\partial \dot{\psi}} = 1 \end{cases}$$

we obtain:

$$\frac{d}{dt}\left(I_1\omega_1\frac{\partial \omega_1}{\partial \dot{\psi}} + I_2\omega_2\frac{\partial \omega_2}{\partial \dot{\psi}} + I_3\omega_3\frac{\partial \omega_3}{\partial \dot{\psi}}\right) =$$

$$= \frac{d}{dt}(I_3\omega_3) = I_3\dot{\omega}_3,$$

$$\frac{\partial T}{\partial \psi} = I_1\omega_1\overbrace{[\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi]}^{\omega_2} +$$
$$+ I_2\omega_2\underbrace{[-\dot{\varphi}\sin\theta\sin\psi - \dot{\theta}\cos\psi]}_{-\omega_1} =$$

$$= I_1\omega_1\omega_2 - I_2\omega_2\omega_1 = \omega_1\omega_2(I_1 - I_2).$$

Finally, $I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = \underbrace{N_\psi}_{\text{"}N_3}$

$\Uparrow$ Euler's EoM #3, as expected

Note that

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\varphi}}\right) - \dfrac{\partial T}{\partial \varphi} = N_\varphi , \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\theta}}\right) - \dfrac{\partial T}{\partial \theta} = N_\theta \end{cases}$$

will <u>not</u> yield the other two Euler's equations since φ & θ do not correspond to rotations around x- and y-axes in the body frame.

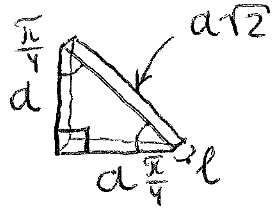
Rather, just guess the other 2 eq's by renaming the axes cyclically:

$3 \to 1, \quad 1 \to 2, \quad 2 \to 3$:

$$\begin{cases} I_1 \dot{\omega}_1 - (I_2 - I_3)\,\omega_2 \omega_3 = N_1 , \\ I_2 \dot{\omega}_2 - (I_3 - I_1)\,\omega_3 \omega_1 = N_2 . \end{cases}$$

This is much easier than trying to extract the eq's directly.

2. $\boxed{5.15}$



mass density per unit volume

triangle mass

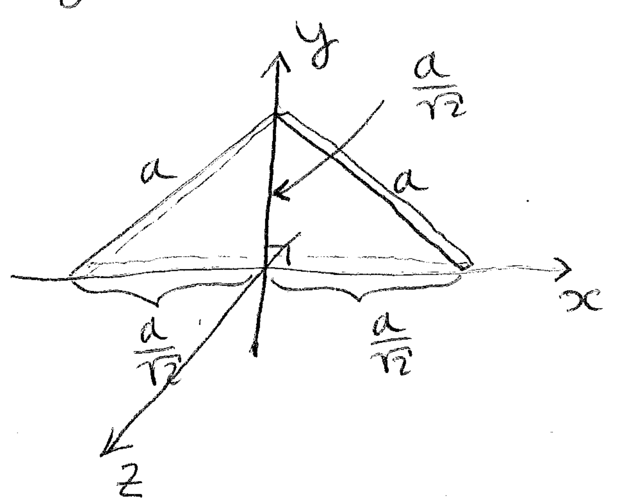
$$\rho V = M$$

$$V = \underbrace{\frac{a^2}{2}}_{A\text{, triangle area}}\ell$$

Note that $\rho\ell = \sigma$, mass density per unit area

$$\sigma A = M$$

Try these axes:



Then $I_{xy} = I_{yx} = -\int dx\,dy\,dz\,\rho\,xy = 0$

by symmetry: for each point $(+x, y)$ there is a corresponding point $(-x, y)$.

Likewise, $I_{xz} = I_{zx} = -\int dx\,dy\,dz\,\rho\,xz = 0$

since $\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} dz\,z = 0$

Finally, $I_{yz} = I_{zy} = -\int_V dx\,dy\,dz\,\rho\,y\,z = 0$

as well since $\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} dz\,z = 0$

So, $I = \begin{pmatrix} I_{xx} & & 0 \\ 0 & I_{yy} & \\ & & I_{zz} \end{pmatrix}$ and the chosen axes are the principal axes

---

Next, compute

$$I_{xx} = \int_V dx\,dy\,dz\,\rho\,\overbrace{(y^2+z^2)}^{\vec{r}^2-x^2} =$$

$$= 2\int_0^{\frac{a}{\sqrt{2}}} dx \int_0^{\frac{a}{\sqrt{2}}-x} dy\,y^2 \times \underbrace{\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} dz\,\rho}_{\sigma} +$$

$$+\rho A \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} dz\,z^2 .$$

$$2\sigma \int_0^{\frac{a}{\sqrt{2}}} dx \int_0^{\frac{a}{\sqrt{2}}-x} dy\,y^2 = \frac{2\sigma}{3}\int_0^{\frac{a}{\sqrt{2}}} dx\left(\frac{a}{\sqrt{2}}-x\right)^3 \;\boxed{=}$$

$u = x - \frac{a}{\sqrt{2}}$

$$\boxed{=}\;\frac{2\sigma}{3}\int_0^{\frac{a}{\sqrt{2}}} du\,u^3 = \frac{Ma^2}{12}$$

$\sigma = \frac{2M}{a^2}$

$$\rho A \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} dz\, z^2 = 2\rho A\left(\frac{\ell}{2}\right)^3 \frac{1}{3} = \frac{\sigma \ell^2}{12} \underbrace{\frac{a^2}{2}}_{A} = \underline{\underline{\frac{M\ell^2}{12}}}.$$

So, $I_{xx} = \frac{M}{12}(a^2+\ell^2)$

Further,

$$I_{yy} = \int_V dx\,dy\,dz\; \rho\,(\overbrace{x^2+z^2}^{\vec{r}^2-y^2}) =$$

$$= \underbrace{2\sigma \int_0^{a/\sqrt{2}} dx \int_0^{\frac{a}{\sqrt{2}}-x} dy\; x^2}_{\shortparallel} + \underbrace{\rho A \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} dz\, z^2}_{\frac{M\ell^2}{12}}.$$

$$2\sigma \int_0^{a/\sqrt{2}} dx\, x^2\left(\frac{a}{\sqrt{2}} - x\right) = 2\sigma\left[\left(\frac{a}{\sqrt{2}}\right)^3 \frac{1}{3}\frac{a}{\sqrt{2}} - \left(\frac{a}{\sqrt{2}}\right)^4 \frac{1}{4}\right] = \underline{\underline{\frac{Ma^2}{12}}}.$$

So, $I_{yy} = \frac{M}{12}(a^2+\ell^2) = I_{xx}$.

Finally, $I_{zz} = \int_V dx\,dy\,dz\,\rho(\overbrace{x^2+y^2}^{\vec{r}^2-z^2}) =$

$$= 2\sigma \int_0^{\frac{a}{\sqrt{2}}} dx \int_0^{\frac{a}{\sqrt{2}}-x} dy\; (x^2+y^2) = \underline{\underline{\frac{Ma^2}{6}}}.$$

Now find CM:

$x_{CM} = 0$, $z_{CM} = 0$ by symmetry

$$y_{CM} = \frac{2\sigma}{M} \int_0^{\frac{a}{\sqrt{2}}} dx \int_0^{\frac{a}{\sqrt{2}}-x} dy\, y = \underline{\underline{\frac{a}{\sqrt{2}\,3}}}.$$

So, $$I_{xx}^{CM} = I_{xx} - M\left(\frac{a}{\sqrt{2}\,3}\right)^2 =$$

$$= \frac{Ma^2}{36} + \frac{M\ell^2}{12},$$

$$I_{yy}^{CM} = I_{yy} = \frac{M(a^2+\ell^2)}{12},$$

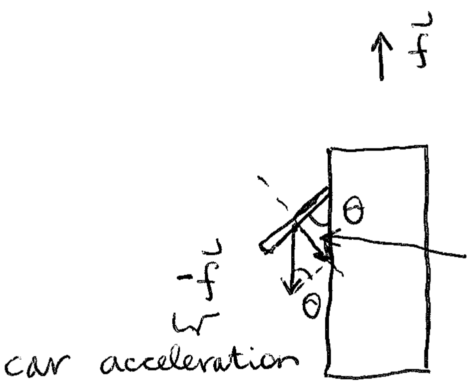
$$I_{zz}^{CM} = I_{zz} - \frac{Ma^2}{18} = \underline{\underline{\frac{Ma^2}{9}}}.$$

(3) $\boxed{5.23}$

EoM: $\underbrace{I}_{\text{moment of inertia}} \ddot{\theta} = a \underbrace{F}_{\text{force}}$

By def'n of the radius of gyration,

$I = m r_0^2$ (we will relate r_0 & a later)



θ = angle between the car & the door

projection of $\vec{f}$ onto the axis $\perp$ the door:

$-f \sin\theta$

The force is $F = -mf\sin\theta$, and

$m r_0^2 \ddot{\theta} = -amf\sin\theta$, or

$$\ddot{\theta} = -\frac{af}{r_0^2}\sin\theta.$$

Now, $\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\underbrace{\frac{d\theta}{dt}}_{\dot{\theta}} = \frac{1}{2}\frac{d}{d\theta}(\dot{\theta}^2)$

Then $\frac{d}{d\theta}(\dot{\theta}^2) = -\frac{2af}{r_0^2}\sin\theta$, yielding

$$\dot{\theta}^2 = \frac{2af}{r_0^2}\cos\theta + \cancel{\text{const}}$$

$$\Downarrow$$

$$\dot{\theta} = \sqrt{\frac{2af}{r_0^2}\cos\theta}$$

Finally, the time it takes for the door to shut:

$$t = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\dot{\theta}} = \int_0^{\frac{\pi}{2}} d\theta \sqrt{\frac{r_0^2}{2af}} \frac{1}{\sqrt{\cos\theta}} \circled{=}$$

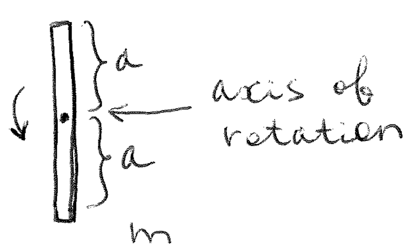
$$\circled{=} \frac{r_0}{\sqrt{2af}} \underbrace{\int_0^{\frac{\pi}{2}} \overbrace{\frac{d\theta}{\sqrt{\cos\theta}}}^{\text{elliptic } \int}}_{\sqrt{2}\,K\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}\,\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}}$$

Now, $t = \dfrac{r_0}{\sqrt{af}} \dfrac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}$

Finally, connect r_0 & a:

$$I' = \rho \int_{-a}^{a} dx\, x^2 = \rho\frac{2a^3}{3} = \frac{ma^2}{3}$$

$$I' = \frac{m}{3}a^2$$



effectively 1D object, with $\rho = \frac{m}{2a}$ linear mass density

Translate to the edge of the rectangle:

$$I = I' + ma^2 = \frac{4}{3}ma^2, \text{ or}$$

$$mr_0^2 \qquad r_0^2 = \frac{4}{3}a^2$$

Finally,

$$t = \sqrt{\frac{a}{f}}\ \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{3\pi}} \approx 3.04\ \text{s}$$