HW#4 solutions

1. $\boxed{4.14}$

(a) In 3D,

$$\varepsilon_{ijp}\,\varepsilon_{rmq} = \begin{vmatrix} \delta_{ir} & \delta_{im} & \delta_{iq} \\ \delta_{jr} & \delta_{jm} & \delta_{jq} \\ \delta_{pr} & \delta_{pm} & \delta_{pq} \end{vmatrix} = \delta_{ir}\delta_{jm}\delta_{pq} + \delta_{im}\delta_{jq}\delta_{pr} + \delta_{jr}\delta_{pm}\delta_{iq} - \delta_{pr}\delta_{jm}\delta_{iq} - \delta_{ir}\delta_{pm}\delta_{jq} - \delta_{jr}\delta_{im}\delta_{pq}.$$

In particular,

$$\underbrace{\varepsilon_{ijp}\,\varepsilon_{rmp}}_{\text{sum over } p} = \delta_{ir}\delta_{jm}\times 3 + \delta_{im}\delta_{jp}\delta_{pr} + \delta_{jr}\delta_{pm}\delta_{ip} - \delta_{pr}\delta_{jm}\delta_{ip} - \delta_{ir}\delta_{pm}\delta_{jp} - \delta_{jr}\delta_{im}\times 3 =$$

$$= 3\delta_{ir}\delta_{jm} + \delta_{im}\delta_{jr} + \delta_{jr}\delta_{im} - \delta_{jm}\delta_{ir} - \delta_{ir}\delta_{jm} - 3\delta_{jr}\delta_{im} = \delta_{ir}\delta_{jm} - \delta_{im}\delta_{jr}.$$

(b) Furthermore,

$$\varepsilon_{ijp}\,\varepsilon_{ijk} = \varepsilon_{pij}\,\varepsilon_{kij} = \delta_{pk}\underbrace{\delta_{ii}}_{3} - \underbrace{\delta_{pi}\delta_{ik}}_{\delta_{pk}} =$$

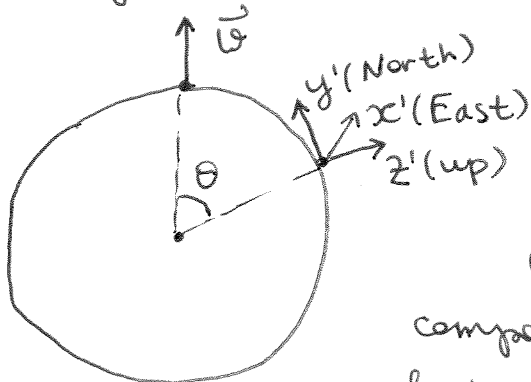
$$= 2\delta_{pk}.$$

(2.) $\boxed{4.22}$

We need to show that

$\underbrace{\psi}_{\text{angular deflection}} = (\omega \cos\theta) \times t$, where $\omega = \omega_{Earth}$ &
θ = co-latitude

The setup is as follows:



A projectile fired horizontally is in the x'y' plane.

Clearly, $\vec{\omega}$ has non-zero components (in general) ω_y & ω_z, but <u>not</u> ω_x. ω_y results in a deflection along the z' axis. Since we're looking for a horizontal deflection, we focus on ω_z.

[The vertical deflection is swamped by gravity anyway]

(↑ in the x'y' plane)

Note that $\omega_z = \omega\cos\theta$, and the magnitude of Coriolis acceleration is

$$a_c = 2v\omega\cos\theta$$

(note that the angle between $\vec{v}$ & $\omega_z \hat{z}'$ is $\frac{\pi}{2}$)

The distance traveled is

$$d = \frac{1}{2} a_c t^2 = v\omega\cos\theta \times t^2.$$

Note that $d=0$ if $\theta = \frac{\pi}{2}$ (equator), since $\omega_z = 0$ in this case.

On the other hand,

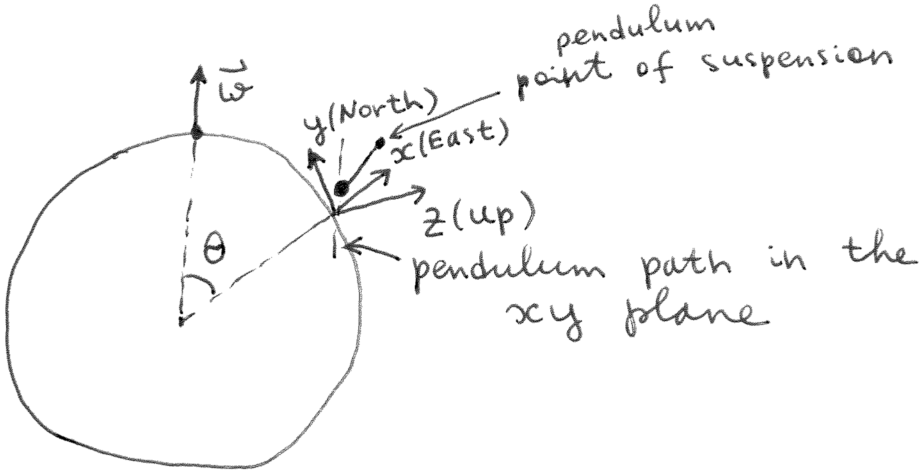
$\underset{\uparrow \atop \psi \ll 1}{d \simeq} \underset{\text{⌞}\! = \! \vartheta t}{x}\, \psi \implies \psi = \frac{d}{x} = \frac{\vartheta \omega \cos\theta \times t^2}{\vartheta t} = \omega \cos\theta \times t,$

as desired.

The direction of deviation can be obtained from $\vec{a}_c = 2(\vec{\vartheta} \times \omega_z \hat{z}')$, to the right of $\vec{\vartheta}$ in the Northern hemisphere.
(i.e. deflecting to the right)

3. $\boxed{4.23}$

Foucault pendulum



If we neglect vertical displacement of the pendulum, its trajectory is traced in the xy plane. As in 4.22, motion in the xy plane is only affected by $\omega_z \hat{z}$, where $\omega_z = \omega\cos\theta$.

Then
$$\text{i.e.}\ \begin{cases} \ddot{x} + \Omega^2 x = 2\omega_z \dot{y}, \\ \ddot{y} + \Omega^2 y = -2\omega_z \dot{x}, \end{cases}$$
where Ω = pendulum frequency.

Using $\xi = x + iy$, we obtain:

$$\ddot{\xi} + \Omega^2 \xi + 2i\omega_z \dot{\xi} = 0 \qquad (*)$$

Since $\omega_z \ll \Omega$, Eq'n $(*)$ can be solved by:

$$\xi(t) = e^{-i\omega_z t}\underbrace{\left[A_1 e^{i\Omega t} + A_2 e^{-i\Omega t} \right]}_{\text{sol'n for a pendulum in inertial frame}} \qquad (**)$$

Indeed, to $\theta(\omega_z)$

$$\begin{cases} \dot{\xi} = (-i\omega_z)\xi + i\Omega e^{-i\omega_z t}\left[A_1 e^{i\Omega t} - A_2 e^{-i\Omega t} \right], \\ \ddot{\xi} \simeq (-i\omega_z)(i\Omega) e^{-i\omega_z t}\left[A_1 e^{i\Omega t} - A_2 e^{-i\Omega t} \right] + \end{cases}$$

$$+ (\Omega\omega_z) e^{-i\omega_z t}\left[A_1 e^{i\Omega t} - A_2 e^{-i\Omega t} \right] +$$

$$+ (i\Omega)^2 e^{-i\omega_z t}\left[A_1 e^{i\Omega t} + A_2 e^{-i\Omega t} \right] =$$

$$= 2(\Omega\omega_z) e^{-i\omega_z t}\left[A_1 e^{i\Omega t} - A_2 e^{-i\Omega t} \right] -$$

$$- \Omega^2 \xi .$$

Now, $\ddot{\xi} + \Omega^2\xi + 2i\omega_z\dot{\xi} \overset{\theta(\omega_z)}{\simeq}$

$$\simeq 2(\Omega\omega_z) e^{-i\omega_z t}\left[A_1 e^{i\Omega t} - A_2 e^{-i\Omega t} \right] -$$

$$- 2(\omega_z\Omega) e^{-i\omega_z t}\left[A_1 e^{i\Omega t} - A_2 e^{-i\Omega t} \right] = 0,$$

as expected

From (**) we see that the ~~plane~~ path in the ~~x~~ xy plane along which the pendulum oscillates rotates with the period

$$T = \frac{2\pi}{\omega\cos\theta} = \frac{T_{Earth}}{\cos\theta}$$

The plane rotates in the clockwise direction in the Northern hemisphere, due to the $e^{-i\omega_z t}$ prefactor in (**).

4. $\boxed{4.1}$

(1) Associativity: $\underline{A(BC) = (AB)C}$.

$$\begin{cases} [A(BC)]_{ij} = A_{ik}(B_{km}C_{mj}), & (1) \\ [(AB)C]_{ij} = (A_{ik}B_{km})C_{mj}. & (2) \end{cases}$$

Clearly, (1) & (2) are identical for $\forall i,j \Rightarrow A(BC) = (AB)C$, as desired.

(2) Orthogonality: $\begin{cases} \tilde{A}A = \mathbb{1}, \\ \tilde{B}B = \mathbb{1} \end{cases}$

Note that $\begin{cases} [\tilde{A}A]_{ij} = \tilde{A}_{ik}A_{kj} = A_{ki}A_{kj} = \delta_{ij}, \\ [\tilde{B}B]_{ij} = B_{ki}B_{kj} = \delta_{ij}. \end{cases}$

But then

$$[(\widetilde{AB})(AB)]_{ij} = (\widetilde{AB})_{ik}(AB)_{kj} = (AB)_{ki}(AB)_{kj} =$$
$$= A_{kl}B_{li}A_{kn}B_{nj} = (\underbrace{A_{kl}A_{kn}}_{\delta_{ln}})(B_{li}B_{nj}) =$$
$$= B_{ni}B_{nj} = \delta_{ij}.$$

Thus, $\widetilde{AB} \times AB = \mathbb{1}$, as desired.

5. $\boxed{4.10}$

(1) $e^B e^C \stackrel{?}{=} e^{B+C}$ if $\underbrace{[B,C]=0}_{BC-CB=0}$

Exponents of matrices are defined by their Taylor expansions. Consider the k^{th} order term of the $e^B e^C$ expansion:

$$\sum_{i=0}^{k} \frac{B^{k-i} C^i}{(k-i)!\, i!} \xleftarrow{(*)} \sum_{\substack{i,j \\ i+j=k}} \frac{B^i}{i!} \frac{C^j}{j!}$$

For example, $k=1$:

$$\underbrace{B}_{i=0} + \underbrace{C}_{i=1}$$, as expected

But $\frac{(B+C)^k}{k!} = \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{(k-j)!\,j!} B^{k-j} C^j = \sum_{j=0}^{k} \frac{B^{k-j} C^j}{(k-j)!\,j!}$, same as (*)

(k^{th} order term of the e^{B+C} expansion) (binomial expansion)

Since k is arbitrary,

this proves $e^B e^C = e^{B+C}$

for commuting B, C.

(2) $A = e^B \overset{?}{\Rightarrow} A^{-1} = e^{-B}$

Consider $A^{-1}A = \mathbb{1}, \quad | \times e^{-B}$

$$A^{-1}(\underbrace{A e^{-B}}) = e^{-B}$$

$$e^B e^{-B} \underset{\uparrow}{=} e^0 = \mathbb{1}$$

$e^B e^C = e^{B+C}$,
$C = -B$ here (note that $[B, -B] = 0$)

This proves $\underline{\underline{A^{-1} = e^{-B}}}$.

(3) $e^{CBC^{-1}} \overset{?}{=} C \underbrace{e^B}_{A} C^{-1}$.

$$e^{CBC^{-1}} = \sum_{n=0}^{\infty} \frac{(CBC^{-1})^n}{n!} =$$

$$= \mathbb{1} + CBC^{-1} + \frac{(CB\overbrace{C^{-1})(C}^{\mathbb{1}}BC^{-1})}{2!} + \ldots =$$

$$= \sum_{n=0}^{\infty} \frac{CB^nC^{-1}}{n!} = C \Big(\underbrace{\sum_{n=0}^{\infty} \frac{B^n}{n!}}_{e^B} \Big) C^{-1} = Ce^BC^{-1},$$

as desired.

(4) If $\underbrace{\tilde{B} = -B}_{B \text{ antisymmetric}} \stackrel{?}{\Rightarrow} \underbrace{\tilde{A} = A^{-1}}_{\substack{A \text{ orthogonal} \\ (A = e^{B})}}$

Consider $\tilde{A} = \widetilde{\left(\sum_{n=0}^{\infty} \frac{B^n}{n!}\right)} = \sum_{n=0}^{\infty} \frac{\widetilde{B^n}}{n!} = \sum_{n=0}^{\infty} \frac{\tilde{B}^n}{n!} \stackrel{\text{(=)}}{}$

use
$\widetilde{XY} = \tilde{Y}\tilde{X}$,
with $\begin{cases} X = B, \\ Y = B^{n-1} \end{cases}$
recursively for $n > 1$

$\stackrel{\text{(=)}}{} \sum_{n=0}^{\infty} \frac{(-B)^n}{n!} = e^{-B}$.

Now recall (2) : $e^{-B} = A^{-1}$ for $A = e^{B}$.

Therefore, $\tilde{A} = e^{-B} = A^{-1}$, proving that A is orthogonal.