

HW #2 solutions

2.10

1. $y = at + bt^2 \Rightarrow \dot{y} = a + 2bt$
 $\leftarrow v = -mgy, y > 0$

Then $\mathcal{I} = \frac{m}{2} \dot{y}^2 + mgy =$ by construction

$$= \frac{m}{2} (a^2 + 4b^2t^2 + 4abt) + mg(at + bt^2) =$$
$$= \frac{ma^2}{2} + ma(2b+g)t + mb(2b+g)t^2, \text{ and}$$

$$\int_0^{t_0} dt \mathcal{I} = \frac{ma^2}{2} t_0 + \frac{m}{2} (2b+g)t_0^2 + \frac{mb}{3} (2b+g)t_0^3.$$

Now, $y_0 = at_0 + bt_0^2$ implies that

$$a = \frac{y_0 - bt_0^2}{t_0}$$

Then $\frac{\partial}{\partial b} \left[\int_0^{t_0} dt \mathcal{I} \right] = mt_0 \frac{y_0 - bt_0^2}{t_0} (-t_0) +$
 $\frac{m}{2} (-t_0)(2b+g)t_0^2 +$

$$+ mt_0^2 \frac{y_0 - bt_0^2}{t_0} + \frac{m}{3} (2b+g)t_0^3 + \frac{2mb}{3} t_0^3 = 0, \text{ or}$$

$$- \frac{m}{6} (2b+g) + \frac{2mb}{3} = 0,$$

$$\frac{b}{3} - \frac{g}{6} = 0 \Rightarrow b = + \frac{g}{2}$$

Finally,

$$a = \frac{y_0 - \frac{g}{2} \left(\frac{2y_0}{g} \right)}{\sqrt{\frac{2y_0}{g}}} = 0, \text{ as desired.}$$

2.

2.12

Consider $\mathcal{L} = \mathcal{L}(q, \dot{q}, \ddot{q}, t)$

for simplicity, focus on a single-particle case

and $I = \int_1^2 dt \mathcal{L}$

Following Goldstein, consider

delta I = integral from 1 to 2 dt { partial L / partial q dq + partial L / partial dot q ddot q + partial L / partial ddot q d^2 q / dt^2 } delta q

integral from 1 to 2 dt partial L / partial dot q d^2 q / dt^2 = partial L / partial dot q [dq/dt]^2 - integral from 1 to 2 dt partial L / partial dot q d/dt (partial L / partial ddot q)

integral from 1 to 2 dt partial L / partial ddot q d^2 q / dt^2 = integral from 1 to 2 dt partial L / partial ddot q d^2/dt^2 (partial L / partial ddot q)

twice by parts, all boundary terms vanish

delta I = integral from 1 to 2 dt { partial L / partial q - d/dt (partial L / partial dot q) + d^2/dt^2 (partial L / partial ddot q) } delta q = 0, yielding

partial L / partial q - d/dt (partial L / partial dot q) + d^2/dt^2 (partial L / partial ddot q) = 0 (*)

Clearly, with n degrees of freedom we have:

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{d^2}{dt^2} \frac{\partial \mathcal{L}}{\partial \ddot{q}_i} = 0, \quad (**)$$

$i=1, \dots, n$

Now, plug

$$\mathcal{L} = -\frac{m}{2} q \ddot{q} - \frac{k}{2} q^2 \quad \text{into } (**):$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial q} = -\frac{m}{2} \ddot{q} - kq, \\ \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0, \\ \frac{\partial \mathcal{L}}{\partial \ddot{q}} = -\frac{m}{2} q \end{cases}$$

$$\Rightarrow -\frac{m}{2} \ddot{q} - kq - \frac{m}{2} \ddot{q} = 0, \text{ or}$$

$$\underline{\underline{m\ddot{q} + kq = 0}}$$

EoM for a harmonic oscillator

So this \mathcal{L} differs from a "normal" \mathcal{L} for an oscillator by $\frac{d}{dt} F$.

$$\mathcal{L}_{\text{sho}} = \frac{m\dot{q}^2}{2} - \frac{kq^2}{2}$$

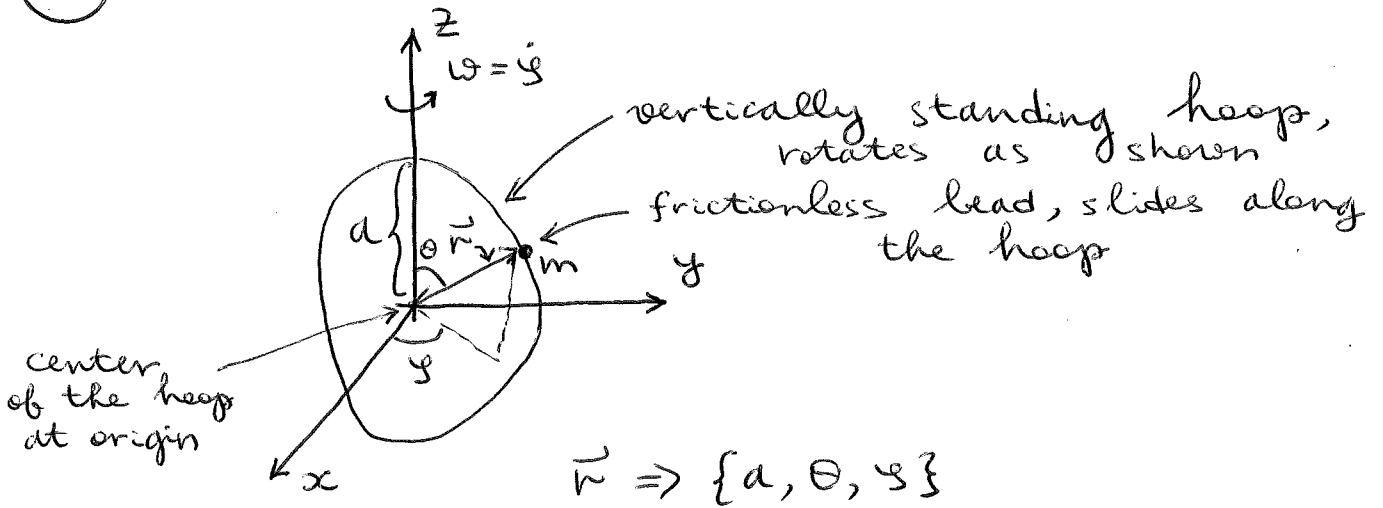
$$\text{Indeed, } \mathcal{L} - \mathcal{L}_{\text{sho}} = -\frac{m}{2} q \ddot{q} - \frac{k}{2} q^2 - \frac{m}{2} \dot{q}^2 + \frac{k}{2} q^2 =$$

$$= -\frac{m}{2} \frac{d}{dt} (q \dot{q}) \Rightarrow \underline{\underline{F = -\frac{m}{2} q \dot{q}}}$$

Thus EoMs obtained using \mathcal{L} & \mathcal{L}_{sho} will be the same.

4.

2.18



More explicitly,

$$\vec{r} = (a \sin \theta \cos \phi, a \sin \theta \sin \phi, a \cos \theta),$$

$$\dot{\vec{r}} = \dot{\vec{r}} = a(\dot{\theta} \cos \theta \cos \phi - \underbrace{\dot{\phi}}_{\omega} \sin \theta \sin \phi, \dot{\theta} \cos \theta \sin \phi + \underbrace{\dot{\phi}}_{\omega} \sin \theta \cos \phi, -\dot{\theta} \sin \theta).$$

Then

$$v^2 = a^2 [\dot{\theta}^2 \cos^2 \theta \cos^2 \phi + \omega^2 \sin^2 \theta \sin^2 \phi - 2\omega \dot{\theta} \sin \theta \cos \theta \sin \phi \cos \phi + \dot{\theta}^2 \cos^2 \theta \sin^2 \phi + \omega^2 \sin^2 \theta \cos^2 \phi + 2\omega \dot{\theta} \sin \theta \cos \theta \sin \phi \cos \phi + \dot{\theta}^2 \sin^2 \theta] =$$

$$= a^2 [\dot{\theta}^2 + \omega^2 \sin^2 \theta].$$

So, $\mathcal{L} = \frac{ma^2}{2} [\dot{\theta}^2 + \omega^2 \sin^2 \theta] - \underbrace{mg a \cos \theta}_{0'' \text{ when } \theta = \frac{\pi}{2}}$

Next, we employ

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \text{ to obtain}$$

$$ma^2 \ddot{\theta} = ma^2 \omega^2 \sin \theta \cos \theta + mga \sin \theta.$$

Clearly, $p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}}$ is not conserved but

$$h = \dot{\theta} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L} \text{ is: } \leftarrow \text{since } \frac{\partial \mathcal{L}}{\partial t} = 0$$

$$h = ma^2 \dot{\theta}^2 - \frac{m}{2} a^2 \dot{\theta}^2 - \frac{ma^2 \omega^2}{2} \sin^2 \theta + mga \cos \theta =$$

$$= \frac{ma^2}{2} \dot{\theta}^2 - \underbrace{\left[\frac{ma^2 \omega^2}{2} \sin^2 \theta - mga \cos \theta \right]}_{V_{\text{eff}}} = \text{const.}$$

We need to find stationary points of V_{eff} :

$$\frac{\partial V_{\text{eff}}}{\partial \theta} = -mga \sin \theta - ma^2 \omega^2 \sin \theta \cos \theta = 0, \text{ or}$$

$$\underbrace{\sin \theta}_{\theta=0, \pi} \underbrace{(g + a\omega^2 \cos \theta)}_{\cos \theta = -\frac{g}{a\omega^2} = -\frac{\omega_0^2}{\omega^2}} = 0.$$

$$\text{define } \omega_0^2 = \frac{g}{a} \text{ [units OK]}$$

Clearly, if $\omega_0 > \omega$ there is no solution
 but for $\omega_0 \leq \omega$ $\theta = \cos^{-1} \left(-\frac{\omega_0^2}{\omega^2} \right)$ becomes a
 valid alternative.

Stability analysis:

divide by
 $ma > 0$
 \downarrow

Consider $\frac{\partial^2 V_{eff}}{\partial \theta^2} \Rightarrow \frac{\partial}{\partial \theta} [-\sin \theta (g + a\omega^2 \cos \theta)] =$

$$= -\cos \theta (g + a\omega^2 \cos \theta) - \sin \theta \cdot a\omega^2 (-\sin \theta) =$$

$$= -g \cos \theta - a\omega^2 \underbrace{\cos 2\theta}_{2 \cos^2 \theta - 1}$$

$\theta = 0$: $-g - a\omega^2 < 0$, always unstable
 [top of the hoop]

$\theta = \pi$: $g - a\omega^2 > 0$ if $\omega^2 < \frac{g}{a} = \omega_0^2$ stable
 [bottom of the hoop] ≤ 0 if $\omega^2 \geq \omega_0^2$ unstable

$$\cos \theta = -\frac{\omega_0^2}{\omega^2}: \frac{g\omega_0^2}{\omega^2} + a\omega^2 \left[1 - 2 \frac{\omega_0^4}{\omega^4} \right] = a \frac{\omega^4 - \omega_0^4}{\omega^2} \geq 0$$

[$\omega \geq \omega_0$] if $\omega^2 \geq \omega_0^2$, stable

So, the stable fixed point will switch from $\theta = \pi$ to $\theta = \cos^{-1} \left(-\frac{\omega_0^2}{\omega^2} \right)$ at $\omega = \omega_0$.

4.

$$\mathcal{L} = \frac{m^2 \dot{x}^4}{12} + m \dot{x}^2 V(x) - V^2(x)$$

$$\text{Thus, } \begin{cases} \frac{\partial \mathcal{L}}{\partial x} = m \dot{x}^2 \frac{dV}{dx} - 2V \frac{dV}{dx}, \\ \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{m^2 \dot{x}^3}{3} + 2m \dot{x} V. \end{cases}$$

$$\text{Next, } \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = m^2 \dot{x}^2 \ddot{x} + 2m \ddot{x} V + 2m \dot{x} \underbrace{\frac{dV}{dt}}_{\dot{x} \frac{dV}{dx}}$$

The EoM is given by

$$m^2 \dot{x}^2 \ddot{x} + 2m \ddot{x} V + 2m \dot{x}^2 \frac{dV}{dx} - m \dot{x}^2 \frac{dV}{dx} + 2V \frac{dV}{dx} = 0, \text{ or}$$

$$(m \ddot{x} + \frac{dV}{dx}) m \dot{x}^2 + (m \ddot{x} + \frac{dV}{dx}) 2V = 0,$$

$$(m \ddot{x} + \frac{dV}{dx})(m \dot{x}^2 + 2V) = 0.$$

$$\text{Introduce } \begin{cases} F(x) = -\frac{dV}{dx}, \\ T = \frac{m \dot{x}^2}{2}. \end{cases}$$

we obtain: $(m \ddot{x} - F)(T + V) = 0.$

So at any time t either $F = m \ddot{x}$ or $T + V = 0.$