1.

(3) Goldstein Ch. 1, Ex. 12 (escape velocity)

For the particle to escape from the Earth's gravitational field, it must, strictly speaking, be at ∞ distance from its surface:

$$V_{(r)} = -G \frac{mM}{R+r}$$

G: gravit'l const; M: Earth's mass; R: Earth's radius; r: distance between particle & Earth's surface

Clearly, $\lim_{r\to\infty} V(r) = 0 \Leftarrow$ "particle escaped"

At that moment, $T=0$ as well, since we are focusing on the minimum escape velocity. But then $E = T+V = 0$ as well.

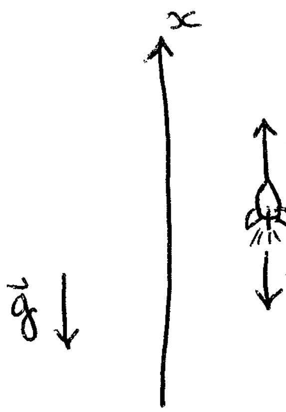
E: total energy

Since total energy is conserved, we have on the Earth surface:

$$\frac{m v_e^2}{2} - G\frac{mM}{R} = 0 \Rightarrow v = \sqrt{\frac{2GM}{R}} \simeq 11.2\ \frac{km}{s}.$$

v_e: escape velocity

2. Goldstein Ch. 1, Ex. 13 (rockets)



time t:

$\vec{v}(t)$ ⇐ velocity

$m(t)$ ⇐ mass

$p(t) = m(t)\,v(t)$ ⇐ momentum

In a small interval Δt, the change in momentum is

$$\Delta p = F\Delta t = -g\,m(t)\,\Delta t$$

During Δt, the rocket releases Δm of gas at velocity v' wrt the rocket.

Thus
$$p(t+\Delta t) = \underbrace{\underbrace{(m(t)-\Delta m)}_{m(t+\Delta t)}\underbrace{(v(t)+\Delta v)}_{v(t+\Delta t)}}_{\text{rocket}} + \underbrace{\Delta m(v(t) - v')}_{\text{gas}}$$

Then
$$\Delta p = p(t+\Delta t) - p(t) \underset{\substack{\text{to } \theta(\Delta m),\\ \theta(\Delta v)}}{=} m(t)\,\Delta v - v'\Delta m = -g\,m(t)\,\Delta t,$$
or

$$\frac{\Delta v}{\Delta t} = -g + \frac{v'}{m(t)}\underbrace{\frac{\Delta m}{\Delta t}}_{\gamma}$$

$$\gamma = \underbrace{\frac{m_0}{60}}_{\equiv\, \alpha m_0,\ \alpha = \frac{1}{60}\,s^{-1}} s^{-1}$$ ⇐ rate of fuel expenditure

(m_0 ⇐ initial mass of the rocket)

Note that
$$\underbrace{\frac{\Delta v}{\Delta t}}_{>0} = -\underbrace{\gamma}_{\substack{>0\\ \text{by def'n}}}\underbrace{\frac{\Delta v}{\Delta m}}_{<0}$$
[$v \nearrow$ as $m \searrow$]

Then $-\gamma \frac{\Delta v}{\Delta m} = -g + \frac{v'}{m}\gamma$, or,

in the $\Delta t \to 0$, $\Delta m \to 0$, $\Delta v \to 0$ limit:

$$dv = \frac{g}{\gamma} dm \mp v' \frac{dm}{m}$$

This yields

$$v(m) = \frac{g}{\gamma}(m - m_0) \mp v' \log\left(\frac{m}{m_0}\right) \qquad (*)$$

$\uparrow$ $v(m_0) = 0$

We need to find $\frac{m_0}{m}$ in terms of v_e, i.e. invert Eq. (*). Using $m \ll m_0$, we obtain:

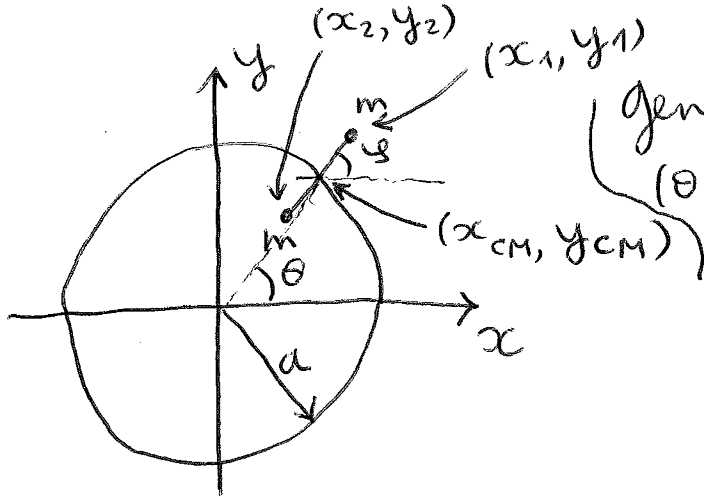
$$v_e = \frac{g}{\alpha}\left(\frac{m}{m_0} - 1\right) \mp |\vec{v}'| \log\left(\frac{m}{m_0}\right) \simeq$$

$$\simeq -\frac{g}{\alpha} + |\vec{v}'| \log\left(\frac{m_0}{m}\right). \qquad (**)$$

Plugging in the numbers and inverting Eq. (**), we obtain:

$\frac{m_0}{m} \simeq 293$, almost 300

3.



Generalized coordinates: (θ, φ)

$$(x_{CM}, y_{CM}) = (a\cos\theta, a\sin\theta)$$

Relative to the CM,

$$\begin{cases} (x_1^{rel}, y_1^{rel}) = \frac{\ell}{2}(\cos\varphi, \sin\varphi), \\ (x_2^{rel}, y_2^{rel}) = -\frac{\ell}{2}(\cos\varphi, \sin\varphi). \end{cases}$$

Then
$$\begin{cases} (x_1, y_1) = (a\cos\theta + \frac{\ell}{2}\cos\varphi, a\sin\theta + \frac{\ell}{2}\sin\varphi), \\ (x_2, y_2) = (a\cos\theta - \frac{\ell}{2}\cos\varphi, a\sin\theta - \frac{\ell}{2}\sin\varphi). \end{cases}$$

Combine the two expressions and drop 1, 2 subscripts:

$$(\dot{x}, \dot{y}) = (-a\sin\theta\,\dot{\theta} \mp \frac{\ell}{2}\sin\varphi\,\dot{\varphi}, \\ a\cos\theta\,\dot{\theta} \pm \frac{\ell}{2}\cos\varphi\,\dot{\varphi}).$$

Then
$$v_{1,2}^2 = \dot{x}^2 + \dot{y}^2 = a^2\dot{\theta}^2 + \frac{\ell^2}{4}\dot{\varphi}^2 \pm a\ell\dot{\theta}\dot{\varphi}\underbrace{(\cos\theta\cos\varphi + \sin\theta\sin\varphi)}_{\cos(\theta-\varphi)}.$$

Finally, $T = \frac{m}{2}(v_1^2 + v_2^2) =$

$$= m\left(a^2\dot{\theta}^2 + \frac{\ell^2}{4}\dot{\varphi}^2\right).$$

(4.) Using $\vec{F} = \dot{\vec{p}}$, we obtain:

$$\vec{F}\cdot\vec{p} = \dot{\vec{p}}\cdot\vec{p} \underset{\vec{p}=m\vec{v}}{=} \Big(\underbrace{\vec{v}\frac{dm}{dt} + m\frac{d\vec{v}}{dt}}_{\frac{d}{dt}(m\vec{v})}\Big)\cdot m\vec{v} \;\textcircled{=}$$

$$\textcircled{=}\; v^2 \underbrace{m\frac{dm}{dt}}_{\frac{1}{2}\frac{d}{dt}m^2} + m^2 \underbrace{\vec{v}\cdot\frac{d\vec{v}}{dt}}_{\frac{1}{2}\frac{d}{dt}v^2} = \frac{1}{2}\frac{d}{dt}(m^2v^2) =$$

$$= \frac{d}{dt}(mT).$$