Final solutions

1. 25

For the skyhook to remain in orbit, the total centripetal force must balance the total gravitational force.

For an infinitesimal element of mass $dm = \rho\, dr$, the centripetal force is

$dm \times r\,\omega^2$ & the gravitational force is

$dm \times \frac{GM}{r^2}$.

Equating the total forces, we obtain:

$$\int_{R_E}^{R_E+\ell} dr\,\rho\, r\,\omega^2 = \int_{R_E}^{R_E+\ell} dr\,\rho\,\frac{GM}{r^2}, \quad \text{or}$$

$$\omega^2\left[\frac{(R_E+\ell)^2}{2} - \frac{R_E^2}{2}\right] = GM\left[\frac{1}{R_E} - \frac{1}{R_E+\ell}\right].$$

We obtain:

$$\frac{2GM}{\omega^2} = \left[(R_E+\ell)^2 - R_E^2\right]\left(\frac{\ell}{R_E(R_E+\ell)}\right)^{-1} =$$

$$= R_E(R_E+\ell)(2R_E+\ell).$$

Solving in terms of ℓ, we obtain:

$$\ell^2 + 3R_E \ell + 2R_E^2 - \frac{2GM}{R_E \omega^2} = 0 \quad \text{, or}$$

$$\ell_{1,2} = \frac{-3R_E \pm \sqrt{9R_E^2 + 4\left(\frac{2GM}{R_E\omega^2} - 2R_E^2\right)}}{2}$$

Only the "+" solution is physical, so that

$$\ell = -\frac{3R_E}{2} + \frac{1}{2}\sqrt{R_E^2 + \frac{8GM}{R_E \omega^2}} = -\frac{3R_E}{2} + \frac{1}{2}\sqrt{R_E\left(R_E + \frac{8g}{\omega^2}\right)}$$

Using
$$\begin{cases} R_E = 6.4 \times 10^6 \text{ m}, \\ \omega = 7.3 \times 10^{-5} \text{ s}^{-1}, \\ \frac{GM}{R_E^2} = g = 9.8 \frac{\text{m}}{\text{s}^2}, \end{cases}$$

we get

$$\ell \simeq 1.44 \times 10^8 \text{ m} = \underline{\underline{1.44 \times 10^5 \text{ km}}}.$$

Finally, note that as $\omega \uparrow$ the $\sqrt{\dots}$ becomes smaller and as $\omega \to +\infty$ there are no valid solutions. Hence there is a critical value of ω above which the skyhook is impossible:

$$R_E\left(R_E + \frac{8g}{\omega^2_{crit}}\right) = 9R_E^2 \Rightarrow \omega_{crit} = \sqrt{\frac{g}{R_E}} \simeq 1.2 \times 10^{-3} \text{ s}^{-1}$$

Luckily, $\omega < \omega_{crit}$ on Earth!

2. Start with 25

$$\mathcal{L} = \underbrace{\frac{m}{2}\left(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\varphi^2\right)}_{T \text{ in spherical coords}} + \underbrace{mgr\cos\theta}_{V = -mgr\cos\theta}.$$

Here, $r = \ell = \text{const} \Rightarrow \dot r = 0$.

Then
$$\begin{cases} p_r = \dfrac{\partial \mathcal{L}}{\partial \dot r} = m\dot r = 0, \\[2mm] p_\theta = \dfrac{\partial \mathcal{L}}{\partial \dot\theta} = mr^2\dot\theta = m\ell^2\dot\theta, \\[2mm] p_\varphi = \dfrac{\partial \mathcal{L}}{\partial \dot\varphi} = mr^2\sin^2\theta\,\dot\varphi = m\ell^2\sin^2\theta\,\dot\varphi \end{cases}$$

Note that φ is cyclic $\left(\dfrac{\partial \mathcal{L}}{\partial \varphi} = 0\right) \Rightarrow \underline{\underline{p_\varphi = \text{const}}}$

Now,
$$H = \underbrace{\cancel{\frac{p_r^2}{2m}}}_{0} + \frac{p_\theta^2}{2m\ell^2} + \frac{p_\varphi^2}{2m\ell^2\sin^2\theta} - mg\ell\cos\theta.$$

<u>EoM</u>: $\dot\theta = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{m\ell^2}$, same as above

Next, $\dot p_\theta = -\dfrac{\partial H}{\partial \theta} = \dfrac{p_\varphi^2}{m\ell^2}\,\dfrac{\cos\theta}{\sin^3\theta} - mg\ell\sin\theta$

At equilibrium, $\dot{p}_\theta = 0$, and

$$\frac{p_\varphi^2}{m^2 g \ell^3} \underbrace{\cos\theta_0}_{\substack{\text{equil.} \\ \text{value of uniform} \\ \text{circular motion}}} = \sin^4\theta_0 \quad \Rightarrow \quad \underline{\underline{p_\varphi = \pm \sin^2\theta_0 \times m \sqrt{\frac{g\ell^3}{\cos\theta_0}}}} \, .$$

Expand around θ_0: $\theta = \theta_0 + \eta$, $\eta \ll 1$

$$H(\theta) \simeq H(\theta_0) + \eta \underbrace{\left.\frac{\partial H}{\partial \theta}\right|_{\theta_0}}_{\text{"0 at equil.}} + \frac{\eta^2}{2} \left.\frac{\partial^2 H}{\partial \theta^2}\right|_{\theta_0}$$

$$\frac{\partial^2 H}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left[mg\ell \sin\theta - \frac{p_\varphi^2}{m\ell^2}\frac{\cos\theta}{\sin^3\theta} \right] =$$

$$= mg\ell\cos\theta - \frac{p_\varphi^2}{m\ell^2}\frac{-\sin^4\theta - 3\sin^2\theta\cos^2\theta}{\sin^6\theta} =$$

$$= mg\ell\cos\theta + \frac{p_\varphi^2}{m\ell^2}\frac{\sin^2\theta + 3\cos^2\theta}{\sin^4\theta} \, .$$

Now,

$$\left.\frac{\partial^2 H}{\partial \theta^2}\right|_{\theta_0} = mg\ell\cos\theta_0 + \underbrace{mg\ell\frac{\sin^4\theta_0}{\cos\theta_0}}_{\frac{p_\varphi^2}{m\ell^2}} \frac{\sin^2\theta_0 + 3\cos^2\theta_0}{\sin^4\theta_0} =$$

$$= mg\ell\cos\theta_0 + mg\ell\frac{1 + 2\cos^2\theta_0}{\cos\theta_0} =$$

$$\stackrel{(=)}{} \underbrace{mgl \frac{1+3\cos^2\theta_0}{\cos\theta_0}}.$$

Note that

$$\left.\frac{\partial^2 H}{\partial \theta^2}\right|_{\theta_0} > 0 \quad \text{for} \quad 0<\theta_0<\frac{\pi}{2},$$

s.t. the circular orbit is stable

Finally,

$$H(\theta) = \underbrace{\frac{p_\theta^2}{2ml^2} + \frac{p_\varphi^2}{2ml^2\sin^2\theta_0} - mgl\cos\theta_0}_{H(\theta_0)} +$$

$$+ \frac{\eta^2}{2}\frac{mgl}{\cos\theta_0}(1+3\cos^2\theta_0) =$$

$$= \underbrace{\frac{p_\theta^2}{2ml^2} + \frac{\eta^2}{2}\frac{mgl}{\cos\theta_0}(1+3\cos^2\theta_0)}_{\text{harmonic oscillator Hamiltonian}} + \text{const}.$$

<u>EoM</u>:

$$\begin{cases} \dot{p}_\theta = -\dfrac{\partial H}{\partial \eta} = -\eta\dfrac{mgl}{\cos\theta_0}(1+3\cos^2\theta_0), \\ \dot{\eta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{ml^2}. \end{cases}$$

Clearly,

$$\ddot{\eta} = \frac{\dot{p}_\theta}{ml^2} = -\eta \underbrace{\frac{g}{l\cos\theta_0}(1+3\cos^2\theta_0)}_{\equiv\, \omega^2(\theta_0)}, \quad \text{or}$$

$$\ddot{\eta} + \omega^2(\theta_0)\eta = 0.$$

This identifies $\omega(\theta_0)$ as the frequency of harmonic motion around a stable circular orbit

3. (a) Kinetic energy: (25)

$$T = \frac{m}{2}\left(\dot{\eta}_1^2 + \dot{\eta}_2^2\right)$$

potential energy:

gravity

$$\eta_i = \ell \sin\theta_i \simeq \ell\theta_i$$

Then $V_g = \frac{mg}{2\ell}\left(\eta_1^2 + \eta_2^2\right)$

$$\ell(1-\cos\theta_i) \simeq \frac{\ell\theta_i^2}{2} \simeq \frac{\eta_i^2}{2\ell}$$

$d = (d_0 + \eta_2) - \eta_1$

Spring $V_s = \frac{k}{2}(d-d_0)^2 = \frac{k}{2}(\eta_1 - \eta_2)^2$

Finally, $V = V_g + V_s = \frac{mg}{2\ell}\left(\eta_1^2 + \eta_2^2\right) + \frac{k}{2}(\eta_1 - \eta_2)^2$

(b) $\mathcal{L} = T - V = \frac{m}{2}\left(\dot{\eta}_1^2 + \dot{\eta}_2^2\right) - \frac{mg}{2\ell}\left(\eta_1^2 + \eta_2^2\right) -$

$- \frac{k}{2}\left(\eta_1^2 - 2\eta_1\eta_2 + \eta_2^2\right)$

EoM: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\eta}_i} = \frac{\partial \mathcal{L}}{\partial \eta_i}$ yield

$$\begin{cases} m\ddot{\eta}_1 = -\left(\frac{mg}{\ell} + k\right)\eta_1 + k\eta_2, \\ m\ddot{\eta}_2 = -\left(\frac{mg}{\ell} + k\right)\eta_2 + k\eta_1 \end{cases}$$

(c) Normal modes: $\eta_i = p_i e^{i\omega t}$ $\quad i=1,2$

$$\begin{cases} \left[-m\omega^2 + \left(\frac{mg}{\ell} + k\right)\right] p_1 - k p_2 = 0, \\ \left[-m\omega^2 + \left(\frac{mg}{\ell} + k\right)\right] p_2 - k p_1 = 0. \end{cases}$$

det $= 0$: Define $\omega_0^2 = \frac{g}{\ell} + \frac{k}{m}$, then

$$(-m\omega^2 + m\omega_0^2)^2 - k^2 = 0,$$

$$\omega_\pm^2 = \omega_0^2 \pm \frac{k}{m} = \begin{cases} \frac{g}{\ell} + \frac{2k}{m}, \\ \frac{g}{\ell} \end{cases}$$

(d) Now, consider

$$\begin{pmatrix} -m\omega_+^2 + m\omega_0^2 & -k \\ -k & \underbrace{-m\omega_+^2 + m\omega_0^2}_{-\frac{mg}{\ell} - 2k + \frac{mg}{\ell} + k \;=\; -k} \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_2^+ \end{pmatrix} = 0, \text{ or}$$

$$\begin{pmatrix} k & k \\ k & k \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_2^+ \end{pmatrix} = 0 \quad \Rightarrow \quad p_1^+ = -p_2^+$$

Likewise,

$$\begin{pmatrix} -m\omega_-^2 + m\omega_0^2 & -k \\ -k & \underbrace{-m\omega_-^2 + m\omega_0^2}_{-\frac{mg}{\ell} + \frac{mg}{\ell} + k = k} \end{pmatrix} \begin{pmatrix} p_1^- \\ p_2^- \end{pmatrix} = 0, \text{ or}$$

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} p_1^- \\ p_2^- \end{pmatrix} = 0 \quad \Rightarrow \quad p_1^- = p_2^-$$

Now,

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = A \cos(\omega_+ t + \varphi_1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} +$$

$$+ B \cos(\omega_- t + \varphi_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{where}$$

A, B & φ_1, φ_2 are determined by ICs.

The system is at rest at $t=0$:

$$\varphi_1 = 0, \quad \varphi_2 = 0$$

$$\begin{pmatrix} \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} 0.1\, d_0 \\ 0 \end{pmatrix} = \begin{pmatrix} A + B \\ B - A \end{pmatrix} \quad \Rightarrow \quad A = B = 0.05\, d_0$$

Thus

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = (0.05 d_0) \cos(\omega_+ t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} +$$

$$+ (0.05 d_0) \cos(\omega_- t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that ω_+ corresponds to the two pendulums moving out of phase; with ω_- the pendulums move in phase and the spring is not stretched (note that ω_- is indep. of k)

4. (a) In cylindrical coordinates,

25

$$\dot{q}_1^2 + \dot{q}_2^2 = \dot{\rho}^2 + \rho^2 \dot{\varphi}^2$$

Hence,

$$\mathcal{L} = T - V = \frac{\rho^2}{2}\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right) - \frac{1}{\rho^2}$$

(b) φ is cyclic $\Rightarrow$ $p_\varphi = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \rho^4 \dot{\varphi}$ is a const of the motion

also, $\dfrac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \underbrace{E}_{\text{total energy}} = \text{const}.$

(c)
$$\begin{cases} p_\rho = \dfrac{\partial \mathcal{L}}{\partial \dot{\rho}} = \rho^2 \dot{\rho}, \\ p_\varphi = \rho^4 \dot{\varphi} \end{cases} \Leftarrow \text{canonical momenta}$$

Then

$$H = p_\rho \dot{\rho} + p_\varphi \dot{\varphi} - \mathcal{L} = \frac{p_\rho^2}{\rho^2} + \frac{p_\varphi^2}{\rho^4} - \frac{\rho^2}{2}\left(\frac{p_\rho^2}{\rho^4} + \rho^2 \frac{p_\varphi^2}{\rho^8}\right) + \frac{1}{\rho^2} = \underbrace{\frac{p_\rho^2}{2\rho^2} + \frac{p_\varphi^2}{2\rho^4} + \frac{1}{\rho^2}}_{H(p_\rho,\, p_\varphi,\, \rho)}.$$

(d) HJ eq'n:

$$(*) \quad H\Big(\rho, \underbrace{\frac{\partial S}{\partial \rho}}_{p_\rho}, \underbrace{\frac{\partial S}{\partial \varphi}}_{p_\varphi}\Big) + \frac{\partial S}{\partial t} = 0, \quad \text{where}$$

$$S = S(\rho, \varphi, \underbrace{\alpha_1, \alpha_2}_{\text{const}}, t)$$

Since $E = \text{const}$ and φ is cyclic, we should try

$$S = W_0(p) + W_1(\varphi) - \underbrace{E}_{=\alpha_2}t .$$

Moreover, since $\frac{\partial S}{\partial \varphi} = \frac{dW_1}{d\varphi} = p_\varphi = \text{const}$,

we have: $W_1^{(\varphi)} = p_\varphi \varphi$ [+ const which we throw out since we only need partial derivatives of S]

In other words, $W_1(\varphi) = p_\varphi \varphi$ is the only form which will leave H indep. of φ.

Now plug S into Eq. (*):

$$\frac{1}{2p^2}\left(\frac{dW_0}{dp}\right)^2 + \frac{p_\varphi^2}{2p^4} + \frac{1}{p^2} = E \quad , \text{ or}$$

$$\frac{dW_0}{dp} = \pm\sqrt{2Ep^2 - \frac{p_\varphi^2}{p^2} - 2} .$$

Now, consider

$$\underbrace{\beta}_{\substack{= \text{const} \\ \text{(new coord.)}}} = \frac{\partial S}{\partial E} = \frac{\partial W_0}{\partial E} - t$$

Finally,

$$W_{0}(p) = \pm \int_{p_0}^{p} dp' \sqrt{2Ep'^2 - \frac{P_\varphi^2}{p'^2} - 2} \quad , \quad \text{yielding}$$

$$\frac{\partial W_0}{\partial E} = \pm \int_{p_0}^{p} dp \frac{2p'^2}{2\sqrt{2Ep'^2 - \frac{P_\varphi^2}{p'^2} - 2}} .$$

We have

$$\boxed{\pm \int_{p_0}^{p} dp \frac{p'^3}{\sqrt{\underbrace{2Ep'^4 - 2p'^2}_{2p'^2(Ep'^2-1)} - P_\varphi^2}} = t + \beta .}$$

This is the implicit equation for $p(t)$ (or, rather, $t(p)$). The equation depends on 2 const of motion: E & P_φ and on 2 const of integration: $\underbrace{p_0}_{= p(t=0)}$ & β.

This is to be expected in a problem with $n=2$.