Symplectic approach to canonical transformations

Consider \[
\begin{cases}
Q_i = Q_i(p, q) & \text{restricted canonical transform} \\
p_i = p_i(p, q) & (*)
\end{cases}
\]

Recall that \(H(q, p, t)\) is obtained from \(H(q, p, t)\) by substituting (i.e., it does not change under restricted transforms)

\[
\begin{cases}
q_j = q_j(p, Q) & (**) \\
p_j = p_j(p, Q)
\end{cases}
\]

Then
\[
\dot{q}_i = \frac{\partial q_i}{\partial q_j} \dot{q}_j + \frac{\partial q_i}{\partial p_j} \dot{p}_j = \frac{\partial q_i}{\partial q_j} \frac{\partial H}{\partial q_j} - \frac{\partial q_i}{\partial p_j} \frac{\partial H}{\partial p_j}
\]

On the other hand,
\[
\frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial q_j} \frac{\partial q_j}{\partial p_i} + \frac{\partial H}{\partial p_j} \frac{\partial q_j}{\partial p_i}.
\]

Thus,
\[
\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \Rightarrow \quad \begin{cases}
(\frac{\partial q_i}{\partial q_j})_{q,p} = \left(\frac{\partial q_i}{\partial p_j}\right)_{q,p} \\
(\frac{\partial q_i}{\partial p_j})_{q,p} = -(\frac{\partial q_j}{\partial p_i})_{q,p}
\end{cases}
\]

Similarly, use \(\dot{p}_i = -\frac{\partial H}{\partial q_i}\) to obtain:

\[
\begin{cases}
(\frac{\partial p_i}{\partial q_j})_{q,p} = -(\frac{\partial q_j}{\partial q_i})_{q,p} \\
(\frac{\partial p_i}{\partial p_j})_{q,p} = (\frac{\partial q_j}{\partial q_i})_{q,p}
\end{cases}
\]
Now use symplectic notation:
\[ \dot{\eta} = J \frac{\partial H}{\partial \eta} \leq H = H(\vec{\eta}, t) \]

Consider a restricted canonical transform
\[ \vec{\xi} = \vec{\xi}(\vec{\eta}) \]

Now,
\[ \dot{\xi}_i = \frac{\partial \xi_i}{\partial \eta_j} \dot{\eta}_j \Rightarrow \dot{\vec{\xi}} = M \dot{\vec{\eta}} \]
\[ \text{Jacobian matrix} \]

Next,
\[ \dot{\vec{\xi}} = MJ \frac{\partial H}{\partial \vec{\eta}} \]

As we change vars in \( H, H = H(\vec{\xi}, t) \),
we obtain:
\[ \frac{\partial H}{\partial \vec{\eta}} = \frac{\partial H}{\partial \xi} \frac{\partial \xi_i}{\partial \eta_j} \Rightarrow M_{ij} = (\tilde{M})_{ij} \]

\[ \frac{\partial H}{\partial \vec{\eta}} = \tilde{M} \frac{\partial H}{\partial \vec{\xi}} \]

Finally,
\[ \dot{\vec{\xi}} = MJ \tilde{M} \frac{\partial H}{\partial \vec{\xi}} \]
\( J \) since the transform is canonical

So,
\[ MJ \tilde{M} = J \]
\[ \Rightarrow MJ = J \tilde{M}^{-1}, \quad \text{or} \quad J(MJ)(-J) = (J(\tilde{M}^{-1})(-J) = JM = \tilde{M}^{-1}J. \]
Finally, \( \vec{M} J \vec{M} = J \) \hspace{1cm} (2)

(1) & (2) are symplectic conditions for a canonical transform.

Ex. Consider \( \vec{\eta} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ p_1 \\ p_2 \end{pmatrix} \) and \( \vec{\xi} = \begin{pmatrix} \dot{Q}_1 \\ Q_2 \\ \dot{P}_1 \\ P_2 \end{pmatrix} \)

and the generating function
\[ F = q_1 p_1 + q_2 p_2 \]

leads to
\[ \begin{cases} Q_1 = q_1, & P_1 = p_1 \\ Q_2 = p_2, & P_2 = -q_2 \end{cases} \]

Then
\[ \begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ p_1 \\ p_2 \end{pmatrix} \]

On the other hand, \( \vec{\eta} = J \frac{\partial H}{\partial \vec{\xi}} \) yields
\[ \begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial H / \partial q_1 \\ \partial H / \partial q_2 \\ \partial H / \partial p_1 \\ \partial H / \partial p_2 \end{pmatrix} \]

It's easy to check that (1) & (2) are satisfied.
Now consider \( \hat{\nu} = \hat{\nu}(\tilde{\eta}, t) \) and focus first on an infinitesimal canonical transform: \( \hat{\nu} = \tilde{\eta} + \delta \tilde{\eta} \Rightarrow \begin{cases} Q_i = q_i \delta + \delta q_i, \\ P_i = p_i + \delta p_i \end{cases} \)

The generating function is \( F_2 = q_i P_i + \delta \in G(q, P, t) \). \( \uparrow \) small pm identity transform

Then \( \begin{cases} P_j = \frac{\partial F_2}{\partial q_j} = P_j + \varepsilon \frac{\partial G}{\partial q_j} \Rightarrow \delta P_j = -\varepsilon \frac{\partial G}{\partial q_j} \\ Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \varepsilon \frac{\partial G}{\partial P_j} \Rightarrow \delta q_j = \varepsilon \frac{\partial G}{\partial P_j} \end{cases} \) to \( \Theta(\varepsilon) \), simply replace \( \{ P \} \) by \( \{ P \} \) everywhere in \( G \) & replace \( \frac{\partial G}{\partial P_j} \rightarrow \frac{\partial G}{\partial P_i} \)

\( G = G(q, P, t) \)

Finally, \( \delta \tilde{\eta} = \varepsilon J \frac{\partial G}{\partial \tilde{\eta}} \) and

\[
M = \frac{\partial \hat{\nu}}{\partial \tilde{\eta}} = I + \varepsilon \frac{\partial \tilde{\eta}}{\partial \tilde{\eta}} \uparrow \begin{array}{c}
\delta \tilde{\eta} = \varepsilon J \frac{\partial G}{\partial \tilde{\eta}} \\
\text{symm. matrix,} \\
\text{if element:} \\
\frac{\partial^2 G}{\partial \tilde{\eta} \partial \tilde{\eta}}
\end{array}
\]
Next, \( \tilde{M} = I + \varepsilon \left( \frac{\partial^2 G}{\partial \eta \partial \eta} \right) J \).

But then

\[
\tilde{M} J \tilde{M} = \left( I + \varepsilon J \frac{\partial^2 G}{\partial \eta \partial \eta} \right) J \left( I - \varepsilon \frac{\partial^2 G}{\partial \eta \partial \eta} J \right) = J.
\]

\[\Theta(\varepsilon)\]

Finally, the reasoning above applies to

\[ \tilde{\xi}_0(t_0) \to \tilde{\xi}_0(t_0 + \delta t) \]

acts as \( \varepsilon \).

Then \( \tilde{\xi}_0(t_0) \to \tilde{\xi}_0(t) \) also satisfies the symplectic condition if we build it up in steps of \( \delta t \).

But the transformation \( \tilde{\eta} \to \tilde{\xi}_0(t_0) \) is

symplectic since it is time-independent.

So, if both \( \tilde{\eta} \to \tilde{\xi}_0(t_0) \) & \( \tilde{\xi}_0(t_0) \to \tilde{\xi}_0(t) \) are

canonical, so is \( \tilde{\eta} \to \tilde{\xi}_0(t) \). Thus any

canonical transform, time-dependent or

not, satisfies the symplectic conditions

(1) & (2).

It can be shown that canonical transformations

form a group, with group multiplication

defined as 2 successive canonical transformations.
Poisson brackets and canonical invariants

Poisson bracket is defined as

\[ \{u, v\}_{\phi, p} = \frac{\partial u}{\partial \phi_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial \phi_i} \]

In matrix form,

\[ \{u, v\}_{\dot{\eta}, \ddot{\eta}} = \frac{\partial u}{\partial \eta} \right\} \cdot \frac{\partial v}{\partial \dot{\eta}} \left( \frac{\partial v}{\partial \ddot{\eta}} \right) \]

Note that

\[
\begin{align*}
\{q_i, q_k\}_{\phi, p} &= \frac{\partial q_i}{\partial \phi_j} \frac{\partial q_k}{\partial p_j} - \frac{\partial q_i}{\partial p_j} \frac{\partial q_k}{\partial \phi_j} = 0, \\
\{p_i, q_k\}_{\phi, p} &= \frac{\partial p_i}{\partial \phi_j} \frac{\partial q_k}{\partial p_j} - \frac{\partial p_i}{\partial p_j} \frac{\partial q_k}{\partial \phi_j} = -\delta_{jk}, \\
\{p_i, p_k\}_{\phi, p} &= 0, \\
\{q_i, p_k\}_{\phi, p} &= \delta_{jk}.
\end{align*}
\]

In matrix form,

\[ \{\ddot{\eta}, \dot{\eta}\}_{\dot{\eta}, \ddot{\eta}} = J. \]

\[ \{\eta, \eta_{m}\}_{\dot{\eta}, \ddot{\eta}} \text{ is the } l, m \text{ element of this matrix} \]
Now consider $\mathcal{I} = \tilde{\mathcal{I}}(\tilde{\eta}, t)$, a time-dependent canonical transformation $(p, q) \to (P, Q)$. In matrix language,

$$\left[ \tilde{\eta} \right] \tilde{\eta} = \frac{\partial \tilde{\eta}}{\partial \eta} J \frac{\partial \tilde{\eta}}{\partial \eta} = \tilde{M} \tilde{J} \tilde{M} = \tilde{J}. \quad (*)$$

$M$, Jacobian matrix

Conversely, if $(*)$ is valid the $\tilde{\eta} \to \tilde{\xi}$ transform must be canonical.

Since $\left[ \tilde{\xi}, \tilde{\xi} \right]_n = \tilde{J}$, $(*)$ implies that Poisson brackets of canonical variables themselves (called fundamental PBS) are invariant under canonical transformations. This is equivalent to $\tilde{M} \tilde{J} \tilde{M} = \tilde{J}$, the symplectic condition of a canonical transform.

Now consider

$$\begin{align*}
\tilde{\eta} & \frac{\partial \tilde{\eta}}{\partial \eta} = \tilde{M} \frac{\partial \eta}{\partial \tilde{\eta}}, \\
\tilde{\xi} & \frac{\partial \tilde{\xi}}{\partial \eta} = \tilde{M} \frac{\partial \eta}{\partial \tilde{\xi}} = \tilde{M} \frac{\partial \eta}{\partial \xi}
\end{align*}$$

Hence

$$\left[ \tilde{u}, \tilde{v} \right] \tilde{\eta} = \frac{\partial \tilde{u}}{\partial \eta} J \frac{\partial \tilde{v}}{\partial \eta} = \frac{\partial \tilde{u}}{\partial \xi} \tilde{M} \frac{\partial \tilde{v}}{\partial \xi} \frac{\partial \tilde{v}}{\partial \xi}

= \left[ \tilde{u}, \tilde{v} \right]_n.$$

So, all Poisson brackets are canonical invariants.
To emphasize that, we shall drop the subscripts: \([u, v]_g \rightarrow [u, v]\) and so on.

Note that \([u, u] = [v, v] = 0\),
\([u, v] = -[v, u]\) (antisymmetry).

Furthermore,
\[
[u + b v, w] = a[u, w] + b[v, w]
\]
(linearity)

\[
[u v, w] = [u, w] v + u [v, w]
\]
(distributive property)

Finally,
\[
[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0
\]
Jacobi's identity

Other canonical invariants:

Schrödinger bracket \([u, v]_{\hbar}\), defined as
\[
[u, v]_{\hbar} = \frac{\partial u}{\partial \tau} \frac{\partial v}{\partial \tau} - \frac{\partial v}{\partial \tau} \frac{\partial u}{\partial \tau}
\]
or
\[
[u, v]_{\hbar} = \frac{\partial u}{\partial \tau} \frac{\partial v}{\partial \tau} - \frac{\partial v}{\partial \tau} \frac{\partial u}{\partial \tau}
\]
in matrix notation

Fundamental Schrödinger brackets:
\[
\begin{cases}
\{q_i, q_j\}_\hbar = 0, & \{p_i, p_j\}_\hbar = 0,
\{q_i, p_j\}_\hbar = \hbar \delta_{ij}
\end{cases}
\]
In matrix notation, \([\vec{\eta}, \vec{\eta}] = J\)
One can use the Jacobian and the symplectic condition to show that \( \{u, v\} \) is canonically im.

Lagrange Poisson brackets are "inverses" to one another, in the following sense:

Consider \( u_i \ (i=1, \ldots, 2n) \), \( 2n \) indep. f's of canonical vars \( q_k \) & \( p_k \) \((k=1, \ldots, n)\).

Then \( \{u, u\} \) is a \( 2n \times 2n \) matrix with

\[ \{u_i, u_j\} \] as the \( ij \)th element. Similarly, \( \{u, u\} \) is a \( 2n \times 2n \) matrix. Then it can be shown that

\[ \{u, u\} [u, u] = -I_{2n}. \]

Lagrange brackets do not obey Jacobi's identity.

Magnitude of a volume element in phase space is canonically im:

Consider \( \eta \to \xi \) [2n-dim phase space]

Then volume element

\[ (d\eta) = dq_1 dq_2 \ldots dq_n dp_1 \ldots dp_n \]

\[ \downarrow \]

\[ (d\xi) = d\theta_1 \ldots d\theta_n dp_1 \ldots dp_n \]
Now, \( (d\%_\phi) = \|M\| (d\eta) \)

absolute value of det of Jacobian matrix

For example, with \( n = 1 \) we have:

\[
\begin{vmatrix}
\frac{\partial \phi}{\partial q} & \frac{\partial \phi}{\partial p} \\
\frac{\partial \eta}{\partial q} & \frac{\partial \eta}{\partial p}
\end{vmatrix}
\]

\( dq dp = [q, p]_\phi \) \( dq dp \)

But the symplectic condition yields:

\(|M|^2 |J| = |J| \Rightarrow |M| = \pm 1, \text{ or} \]

\(|M| = 1. \)

Then \( (d\%_\phi) = (d\eta) \) and therefore \( V_n = \int \cdots \int (d\eta) \) is a canonical invariant volume of an arbitrary region of phase space.

If \( n = 1, \) \( (d\eta) = dq dp \) \& \( V_1 = \int dq dp \)