Lecture 1

Single-particle mechanics

\[ \vec{v} = \frac{d\vec{r}}{dt} \quad \text{velocity} \]

\[ \vec{p} = m\vec{v} \quad \text{linear momentum} \]

\[ \text{part. mass} \]

EoM: (*) \[ \frac{d\vec{p}}{dt} = \vec{F} \quad \text{Newton's 2nd law} \]

\[ \vec{F} \quad \text{total force acting on particle} \]

\[ \vec{p} \quad \text{total momentum} \]

So, \[ \vec{F} = \frac{d}{dt}(m\vec{v}) = m \frac{d\vec{v}}{dt} = ma, \quad \text{where} \]

\[ m = \text{const} \]

\[ a = \frac{d^2\vec{r}}{dt^2} \quad \text{part. acceleration} \]

So, EoM is typically a 2nd order DE in \( \vec{r} \).

(*) is valid in an inertial reference frame.

Conservation laws:

If \( \vec{F} = 0 \), \( \vec{p} = \text{const} \)

Angular momentum: \( L = \vec{r} \times \vec{p} \)

Torque: \( \vec{N} = \vec{r} \times \vec{F} \)

Consider \( \vec{r} \times \vec{F} = \vec{r} \times \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{dL}{dt} \)

\[ \frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{v} \times m\vec{v} + \vec{r} \times \frac{d}{dt}(m\vec{v}) \]

\( = 0 \)
So, if \( N = 0 \), \( \vec{L} = \text{const} \)

Next, consider

\[
W_{12} = \int_1^2 \vec{d} \cdot \vec{F}
\]

Work done on the particle (by def.)

\[
W_{12} = \int_1^2 \vec{d} \cdot \vec{F} = m \int_1^2 \frac{d}{dt} (\vec{v}^2) dt = m \int_1^2 \frac{d}{dt} \left( \frac{m \vec{v}_1^2}{2} - \frac{m \vec{v}_1^2}{2} \right) dt = T_2 - T_1
\]

where \( T = \frac{m \vec{v}_1^2}{2} \) is the particle's kinetic energy.

Note that we used \( \frac{d\vec{d}}{dt} = \vec{\dot{d}} \) here, as well as \( \vec{F}(\vec{r}) = m \vec{a} \) [no dissipation].

If, instead, we use a different path in the force field \( \vec{F}(\vec{r}) \), we again obtain \( W_{12} = T_2 - T_1 \) or, equivalently,

\[
\int_1^2 \vec{F} \cdot d\vec{s} = 0
\]

But then \( \vec{F}(\vec{r}) = -\nabla V(\vec{r}) \), where \( V(\vec{r}) \) is the potential energy.

Indeed, if \( W_{12} \) is independent of the path, we must have \( \vec{F} \cdot d\vec{s} = -dV \) (so that \( \int_1^2 \vec{d} \cdot \vec{F} \) depends only on the endpoints), and thus

\[
Fs = -\frac{\partial V}{\partial s}
\]

Note that \( V(\vec{r}) \) is defined up to a const.
Finally, \( W_{12} = V_1 - V_2 = T_2 - T_1 \), or
\[
\frac{T_1 + V_1}{\text{total E}} = \frac{T_2 + V_2}{\text{energy}} \quad \text{energy conservation (forces must be conservative)}
\]

If \( V = V(\vec{r}, t) \), \( E \) is no longer conserved in general.

\[\sum_{j \neq i} \vec{F}_{ij} + \vec{F}_i^{(e)} = \vec{P}_i \quad \text{J EoM for particle } i\]

Mechanics of a system of particles:

\[\sum_{j \neq i} \text{internal external force exerted by } j \text{th particle} \quad \vec{F}_{ij}, \text{ total external force}\]

Then \( \frac{d^2}{dt^2} \sum_i m_i \vec{\dot{r}}_i = \sum_i \vec{F}_i^{(e)} + \sum_{j \neq i} \vec{F}_{ij} \quad \text{sum over } i \]

\( = 0 \) since \( \vec{F}_{ij} + \vec{F}_{ji} = 0 \), \( \forall i,j \)

by assumption (Newton's 3rd law of motion)

Now, define \( \vec{R} = \vec{\dot{r}} \quad \text{center of mass} \)

\( M \), total mass

Then \( M \frac{d^2 \vec{R}}{dt^2} = \vec{F}^{(e)} \quad \text{J EoM for the center of mass (CoM)} \)

Examples: exploding shell, rocket propulsion
Further, \[ \dot{\mathbf{P}} = \sum_{i} m_i \frac{d\mathbf{r}_i}{dt} = M \frac{d\mathbf{R}}{dt} \]

Total linear momentum

Thus, if \( \dot{\mathbf{F}}^{(e)} = 0 \), \( \dot{\mathbf{P}} = \text{const} \)

\[ \sum_{j \neq i} \mathbf{F}_{ji} + \mathbf{F}_{i}^{(e)} = \sum_{i} \frac{d}{dt} (\mathbf{r}_i \times \mathbf{p}_i) \]

\[ \mathbf{u}_i \times \mathbf{p}_i = 0, \forall i \]

\[ \implies \frac{d}{dt} (\sum_{i} \mathbf{r}_i \times \mathbf{p}_i) \]

Total angular momentum

So, \[ \frac{d\mathbf{L}}{dt} = \sum_{i} \mathbf{r}_i \times \dot{\mathbf{F}}_{i}^{(e)} + \sum_{i, j \neq i} \mathbf{r}_i \times \mathbf{F}_{ji} \]

\( N^{(e)} \) torque

for each pair \( ij \)

\[ \mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ji} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji} \]

Often, \( \mathbf{r}_i \times \mathbf{F}_{ji} = 0 \) (i.e., \( \mathbf{r}_j \uparrow \mathbf{F}_{ji} \uparrow \mathbf{r}_i \))

Then \[ \frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)} \], and

if \( \mathbf{N}^{(e)} = 0 \), \( \mathbf{L} = \text{const} \)
Recall that \( \mathbf{\dot{L}} = \sum_{i} \mathbf{\dot{R}}_i \times \mathbf{p}_i \).

Consider \( \mathbf{\dot{R}}_i = \mathbf{\dot{R}}'_i + \mathbf{\dot{R}} \),

\[ \mathbf{\dot{\mathbf{\dot{R}}}_i} = \mathbf{\dot{\mathbf{\dot{R}}}}'_i + \mathbf{\dot{R}} \]

Then \( \mathbf{\dot{L}} = \sum_{i} \mathbf{R} \times (m_i \mathbf{\dot{\mathbf{\dot{R}}}}_i) + \sum_{i} \mathbf{\dot{R}}'_i \times (m_i \mathbf{\dot{\mathbf{\dot{R}}}}'_i) + \sum_{i} \mathbf{\dot{R}} \times (m_i \mathbf{\dot{\mathbf{\dot{R}}}}'_i) = \]

\[ = \mathbf{R} \times (m \mathbf{\dot{\mathbf{\dot{R}}}}) + \sum_{i} \mathbf{\dot{R}}'_i \times \mathbf{p}'_i + \left( \sum_{i} m_i \mathbf{\dot{\mathbf{\dot{R}}}}'_i \right) \times \mathbf{\dot{\mathbf{\dot{R}}}} = 0 \]

\[ \sum_{i} \mathbf{\dot{R}}'_i \times \mathbf{p}'_i \]

So, \( \mathbf{\dot{L}} = \text{CoM motion} + \text{motion about CoM} \), in general depends on the origin \( \Theta \). However, if \( \mathbf{\dot{v}} = \frac{d\mathbf{R}}{dt} = 0 \) (CoM at rest wrt \( \Theta \)),

\[ \mathbf{\dot{L}} = \sum_{i} \mathbf{\dot{R}}'_i \times \mathbf{p}_i \], angular momentum taken about the CoM.

Finally, let's consider energy:

\[ W_{12} = \sum_{i} \int_{1}^{2} \mathbf{\dot{s}}_i \cdot \mathbf{\dot{F}}_i = \sum_{i} \int_{1}^{2} \mathbf{\dot{v}}_i \cdot \mathbf{\dot{t}} \cdot (m_i \mathbf{\dot{\mathbf{\dot{R}}}}'_i) = \]

\[ = \sum_{i} \int_{1}^{2} \mathbf{\dot{s}}_i \cdot \frac{d}{dt} \left( \frac{m_i \mathbf{\dot{v}}^2}{2} \right) = T_2 - T_1, \] where
\( T = \frac{1}{2} \sum m_i v_i^2 \) is the total kinetic energy.

Note that

\[
T = \frac{1}{2} \sum m_i (\vec{v}_i + \vec{\dot{v}}_i) \cdot (\vec{v}_i + \vec{\dot{v}}_i) = \frac{1}{2} \sum m_i v_i^2 + \frac{1}{2} \sum m_i \dot{v}_i^2 + \vec{v}_i \cdot \frac{d}{dt} \left( \sum m_i \vec{r}_i \right) 
\]

So,

\[
T = \frac{M \dot{r}/2}{2} + \frac{1}{2} \sum m_i \dot{v}_i^2 \quad \text{relative to \text{coM}}
\]

Now,

\[
W_{12} = \sum_i \int_1^2 d\vec{r}_i \cdot \vec{F}_i = \sum_i \int_1^2 d\vec{r}_i \cdot \vec{F}_i^{(e)} + \sum_{i \neq j} \int_i^2 d\vec{r}_j \cdot \vec{F}_{ji} \quad \text{conservative forces:}
\]

\[
\sum_i \int_1^2 d\vec{r}_i \cdot (\vec{v}_i \cdot \nabla_i V_i) = -\nabla_i V_i \quad \text{internal forces equal and opposite}
\]

Indeed,

\[
\int \vec{v}_j \cdot (\vec{r}_i - \vec{r}_j) \frac{1}{|\vec{r}_i - \vec{r}_j|} = \int \vec{v}_j \cdot \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|} = \vec{v}_j \cdot \left( \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|} \right) = -\vec{v}_j \cdot \left( \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|} \right)
\]
\[ \frac{1}{2} \sum_{i \neq j} \sum_{i,j} \nabla_i V_{ij} \cdot d \vec{r}_{ij} = -\sum_i V_i \bigg|_1 - \frac{1}{2} \sum_{i,j} \big| V_{ij} \big|^2 . \]

for each pair of particles, we have

\[ -\int \nabla_i V_{ij} \cdot d \vec{s}_i + \nabla_j V_{ij} \cdot d \vec{s}_j \]

Recall that \( \vec{r}_{ij} = \vec{r}_i - \vec{r}_j \), yielding

\[ \nabla_i V_{ij} = -\nabla_j V_{ij} = \nabla_{ij} V_{ij} \]

wrt \( \vec{r}_i - \vec{r}_j \)

\[ -\int \nabla_i V_{ij} \cdot (d \vec{s}_i - d \vec{s}_j) \]

\[ \frac{d \vec{r}_{ij}}{d \vec{r}_{ij}} \]

The total potential energy is given by

\[ V = \sum_i V_i + \frac{1}{2} \sum_{i,j} \big| V_{ij} \big|^2 , \text{ and} \]

\[ T_2 - T_1 = -V_2 + V_1 \]

\[ T_1 + V_1 = T_2 + V_2 \]

total energy \( T+V \) is conserved