Lecture 14
Symmetrical top with one point fixed

\[ x'y'z': \text{ lab} \]
\[ xyz: \text{ body} \]

\[ \begin{aligned}
\theta & \quad \text{rotation, "bobbing" up and down along } z \text{ (and the top) wrt } z', \\
\phi & \quad \text{precession, rotation of } z \text{ around } z' \text{ (imagine that } \theta \text{ is fixed),} \\
\gamma & \quad \text{rotation of the top around } z \uparrow \text{ body principal axis}
\end{aligned} \]

Often, \( \gamma \gg \theta, \phi \) is in practice
Use the Lagrangian approach:

\[ T = \frac{I_1}{2} (\dot{\omega}_1^2 + \dot{\omega}_2^2) + \frac{I_3}{2} \dot{\omega}_3^2 \]

It can be shown that

\[
\begin{align*}
\omega_1 &= \dot{\theta} \sin \theta \sin \gamma + \dot{\gamma} \cos \theta, \\
\omega_2 &= \dot{\gamma} \sin \theta \cos \gamma - \dot{\theta} \sin \gamma, \\
\omega_3 &= \dot{\gamma} \cos \theta + \dot{\gamma}.
\end{align*}
\]

Then

\[
\begin{align*}
\omega_1^2 + \omega_2^2 &= \dot{\gamma}^2 \sin^2 \theta \sin^2 \gamma + \dot{\theta}^2 \cos^2 \gamma + \\
&+ 2 \dot{\theta} \dot{\gamma} \sin \theta \sin \gamma \cos \gamma + \dot{\gamma}^2 \sin^2 \theta \cos^2 \gamma + \\
&+ \dot{\theta}^2 \sin^2 \gamma - 2 \dot{\theta} \dot{\gamma} \sin \theta \cos \gamma \sin \gamma = \\
&= \dot{\gamma}^2 \sin^2 \theta + \dot{\theta}^2.
\end{align*}
\]

Therefore,

\[
T = \frac{I_1}{2} (\dot{\gamma}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{I_3}{2} (\dot{\gamma} \cos \theta + \dot{\gamma})^2
\]

Furthermore, the potential energy of any body is

\[
V = -\sum \mathbf{m}_i \mathbf{r}_i \mathbf{g} = -\mathbf{M} \mathbf{g} \cdot \mathbf{\dot{r}}.
\]

In this case, \( V = M g l \cos \theta \), and
\[ I = \frac{I_1}{2} (\dot{\theta} + \dot{\gamma}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\gamma} + \dot{\gamma} \cos \theta)^2 - Mg l \cos \theta. \]

Note that \( \dot{\gamma} \) and \( \dot{\gamma} \) are cyclic co-ords:

(1)

\[ p_\gamma = \frac{\partial I}{\partial \dot{\gamma}} = I_3 (\dot{\gamma} + \dot{\gamma} \cos \theta) = I_3 \omega_3 \equiv I_1 a = \text{const}, \]

\( \omega_3 \) yielding \( \omega_3 = \text{const} \)

Next,

(2)

\[ p_\theta = \frac{\partial I}{\partial \dot{\theta}} = (I_1 \sin^2 \theta + I_3 \cos^2 \theta) \dot{\theta} + I_3 \dot{\gamma} \cos \theta \equiv I_1 b = \text{const} \]

Finally,

\[ E = T + V = \frac{I_1}{2} (\dot{\theta}^2 + \dot{\gamma}^2 \sin^2 \theta) + \frac{I_3}{2} \omega_3^2 + M g l \cos \theta = \text{const} = \text{const}. \]

Now,

(1)

\[ I_3 \dot{\gamma} = I_1 a - I_3 \dot{\gamma} \cos \theta \quad \text{and} \]

(2)

\[ I_1 \dot{\theta} \sin^2 \theta + I_1 a \cos \theta = I_1 b, \quad \text{yielding} \]

\[ \dot{\gamma} = \frac{b - a \cos \theta}{\sin^2 \theta}. \]

Thus, if we know \( \theta(t) \), we could find \( \dot{\gamma} \) and then \( \dot{\theta} \):

\[ \dot{\theta} = \frac{b - a \cos \theta}{\sin^2 \theta}. \]
\[ \psi = \frac{I_1}{I_3} a - \frac{6 - a \cos \theta}{\sin^2 \theta} \cos \theta \]

We can get an eq'n for \( \theta \) alone:

\[ E - \frac{I_3}{2} \dot{\theta}^2 = \frac{I_1 \dot{\theta}^2}{2} + \frac{I_1}{2} \frac{(6 - a \cos \theta)^2}{\sin^2 \theta} + Mg \ell \cos \theta \]

\( E' = \text{const} \) \uparrow \quad \text{effective 1D potential} \]

Define \( \psi \) normalized constants:

\[
\begin{align*}
L &= \frac{2E - I_3 \dot{\theta}^2}{I_1}, \\
\beta &= \frac{2Mg \ell}{I_1}, \\
\alpha &= \frac{p_\psi}{I_1}, \\
\beta &= \frac{p_\beta}{I_1}
\end{align*}
\]

Then (*) becomes

\[ L = \dot{\theta}^2 + \frac{(b - a \cos \theta)^2}{\sin^2 \theta} + \beta \cos \theta \]

Next, introduce \( U = \cos \theta \):

\[ \dot{U} = -\sin \theta \dot{\theta}, \quad U^2 = \sin^2 \theta \times \dot{\theta}^2 \]
\[
\frac{2 \sin^2 \theta}{1-u^2} = \frac{\theta}{u} + \frac{(b-a \cos \theta)^2 + \beta \sin^2 \theta \cos \theta}{u} \cdot \frac{1-u^2}{u},
\]

\[
u^2 = (2-\beta u)(1-u^2) - (b-a u)^2 \leq \beta u^3 - (2+a^2) u^2 + (2a \beta - \beta) u + (2-b^2) \equiv f(u)
\]

Note that \( \beta > 0 \), with \( \beta = 0 \) iff \( l = 0 \) ⇒ the fixed point is the CoM, as in gyroscopes. Then \( f(u) \) is quadratic. Here we will focus on \( \beta > 0 \).

\[
f(u) = 0 \quad (i.e. \text{the roots of the cubic eq'n}) \text{ provide information about the turning points.}
\]

Physically, \(-1 \leq u \leq 1\). Moreover, \( \cos \theta \).

If the top is on a horizontal surface, \( u > 0 \) since \( \cos \theta \) is never negative. Then the turning points of \( u \) are the turning points of \( \theta \) since \( \sin \theta \) does not change sign between \( 0, \frac{\pi}{2} \).

Let's formally consider \( f(u) \) in the \((-\infty, +\infty)\) range. Clearly, for \(|u| \gg 1\):

\[
f(u) \approx \beta u^3 \quad [\text{pos. for } u > 0 \text{ & neg. for } u < 0]
\]

Next, focus on \( u = \pm 1 \):
\[ f(1) = \beta - (a + a^2) + (2ab - \beta) + (a - b^2) = - (a - b)^2 < 0. \]

Likewise,

\[ f(-1) = -\beta - (a + a^2) - (2ab - \beta) + (a - b^2) = - (a + b)^2 < 0. \]

The only exception is \( \theta = 0 \) (vertical top) because then

\[
\begin{align*}
\Phi &= I_3 (\dot{\varphi} + \dot{\psi}) = I_1 a, \\
\Psi &= I_3 \dot{\psi} + I_3 \dot{\varphi} = I_1 b
\end{align*}
\]

This means that \( f(1) = 0 \) and \( u = 1 \) is a root. If \( f(1) \) is not a root,

\( f(u) \) qualitatively looks like this:

\[ U_1 \text{ & } U_2 \text{ are located somewhere between } -1 \text{ & } 1; \cos \theta \text{ moves either between } U_1 \text{ & } U_2 \text{ or between } U_{\text{min}} \text{ & } U_2, \text{ where } U_{\text{min}} \text{ is given by the max angle } \Theta_{\text{max}}. \]
allowed by the horizontal surface constraint.

Top's motion is often visualized as a trace its z-axis would have left on a unit sphere centered on a fixed point. This trace is known as the locus of the z-axis. The locus lies between 2 circles: \( \theta_1 = \arccos u_1 \) \& \( \theta_2 = \arccos u_2 \), at which \( \dot{\theta} = 0 \). The shape of the locus curve is largely determined by \( u' = \frac{b}{a} \).

Suppose that \( u' > u_2 \), then

\[
\dot{g} = \frac{b - a \cos \theta}{\sin^2 \theta} = \frac{u' - u}{1 - u^2} a > 0
\]

\( \uparrow \) precession velocity

\( \dot{g} > 0 \) means that \( g \) only increases with time:

In general, the top executes nutation (changes in \( \theta \)) \& precession (changes in \( \dot{\theta} \)) as it rotates around its own z-axis.
If \( u_1 < u' < u_2 \), \( \dot{\theta} > 0 \) at \( u_1 \), but
\( \dot{\theta} < 0 \) at \( u_2 \). In general, \( \dot{\theta} \) does
not vanish on average, so there's
still precession overall:

What if \( u' = u_{21} \)? (for example;
\( u' = u_1 \) is similar)

Then \( \dot{\theta} = 0 \) if \( u = u_{21} \) and \( \dot{\theta} > 0 \) if \( u = u_1 \):

Consider a top spinning around its
\( z \)-axis (inclined at some angle \( \theta_0 \) wrt \( z' \))
at \( t = 0 \): \( \theta = \theta_0 \), \( \dot{\theta} = 0 \), \( \dot{\theta}_z = 0 \) are the
initial conditions.
Since \( \theta = 0 \), \( u = 0 \) and \( u_0 = \cos \theta_0 \) is automatically a root of \( f(u) \).
Since \( \dot{\theta} = 0 \), \( b - d u_0 = 0 \), or
\[
\frac{b}{a} = u_0.
\]

Now, \( E' = E - \frac{I_3}{2} \omega_3^2 = Mgl \cos \theta_0 \) at \( t = 0 \).
As \( t \to \infty \), \( \theta \) and \( \dot{\theta} \) become non-zero, and the corresponding kinetic energy becomes positive. This can only be accomplished if \( \dot{V} \dot{W} \) (i.e., \( \theta \dot{\theta} \)) since the total energy is conserved. Thus \( \theta_0 = \theta_2 \) which corresponds to \( u_0 = \frac{b}{a} \), meaning that
\[
\frac{b}{a} = u_0 = u' = u_2.
\]

In other words, the top always starts to fall after it's released and continues to fall until \( \theta = \theta_1 \), at which point it rebounds back to \( \theta_2 \). Precession is always in one direction; the locus is shown in the last Fig. above.