Review Lecture

We covered material from Goldstein Ch 1-6 and 8-10

Topics that can be on the exam

1. Lagrangian, Lagrange's eqs (Euler–Lagrange eqs) with/without constraints. Cyclic variables.

2. Central force problem. In particular, $\frac{1}{r}$ potential. Scattering from central potential

In principle, any topic covered in lectures can be on the exam.

Best preparation: solve various problems on these topics
1. A particle of mass $m$ is moving in a central potential $V(r)$. As usual we fix the additive constant in the potential so that $V(r) \to 0$ as $r \to \infty$. The particle orbit is given by $r(\theta) = A e^{B \theta}$. This is a logarithmic spiral. Determine $V(r)$. What is the particle’s energy?

\[
\begin{align*}
\frac{m\dot{r}^2}{2} + V(r) + \frac{e^2}{2 \mu r^2} &= E \\
\frac{dr}{d\tau} = \frac{dr}{d\theta} \frac{d\theta}{d\tau} &= \frac{dr}{d\theta} \frac{e}{\mu r^2} = B A e^{B \theta} \frac{e}{\mu r^2} \\
&= \frac{B e}{\mu r} \\
V(r) &= E - \frac{e^2}{2 \mu r^2} - \frac{B^2 e^2}{2 \mu r^2} = E - \frac{(B^2 + 1) e^2}{2 \mu r^2} \\
V(r) \to 0 \quad \text{as} \quad r \to \infty \quad \Rightarrow \quad E = 0 \\
\end{align*}
\]

$x = A \cos \theta \quad \theta = \omega t$
2. A beam of particles of mass $m$ scatters from the following central potential:

$$V(r) = \begin{cases} 
0, & r > a, \\
-V_0, & r \leq a,
\end{cases}$$

where $a$ and $V_0$ are constants. This is a spherical potential well. Assume the particles start at $r \to \infty$ with velocity $v_0$ and let $s$ be the impact parameter. Calculate the scattering angle $\Theta(s)$. What is the total scattering cross section?

$$6(\Theta) = \frac{s}{\sin \Theta} \left| \frac{d\Theta}{d\Theta} \right|$$

$$6_{\text{tot}} = \pi R^2$$

$$\Theta = \pi - 2\psi$$
\[ \Theta = \pi - 2\Phi \]

\[ \Phi = \phi + \beta \]

\[ s = \sin^{-1} \frac{s}{a} \]

\[ \beta = \cos^{-1} \frac{d}{a} \]

\[ \frac{m v_0^2}{2} = \frac{u v^2}{2} - v_0 \]

\[ \gamma = \sqrt{\frac{v_0^2}{w}} \]

\[ m v d = m v_0 s \]

\[ d = s \frac{v_0}{v} = \frac{s}{\sqrt{1 + \frac{2V_0}{m v_0^2}}} \]
\( C_4 = \pi - 2 \sin^{-1} \frac{s}{a} - 2 \cos^{-1} \frac{d}{a} \)

\[ d = \frac{s}{\sqrt{1 + \frac{2V_0}{m \sigma_0^2}}} \]
3. A homogeneous cube, each edge of which has a length $a$, is initially in a position of an unstable equilibrium with one edge in contact with a *frictionless* horizontal plane as shown in the figure below. The cube is then given a small displacement and allowed to fall. Find the angular velocity $\omega$ of the cube at the instant it strikes the plane.

\[ \omega \left( \frac{a}{\sqrt{2}} - \frac{a}{2} \right) = T = \frac{I \omega^2}{2} + \frac{m v_r^2}{2} \]
\[
\begin{align*}
\omega^2 &= \frac{2y}{5} \left( \frac{1}{\sqrt{2}} - \frac{1}{2} \right) \frac{8}{a} \\
\omega^2 &= \frac{12(\sqrt{2} - 1)}{5} \frac{8}{a}
\end{align*}
\]
\[ I = \pi \int_0^{a/2} \int_0^{a/2} \rho (x^2 + y^2) \, dx \, dy = \]
\[ = 8 \rho a \int_0^{a/2} dx \int_0^{a/2} x^2 \, dy = \]
\[ = 8 \rho a \left( \frac{a}{2} \right)^3 \frac{1}{3} \frac{a}{2} = \frac{\rho a^3 a^2}{6} = \]
\[ = \frac{ma^2}{6} \]
\[
\int_{0}^{a/2} \int_{0}^{a/2} \rho \left( x^2 + y^2 \right) \, dx \, dy = \int_{0}^{a/2} \int_{0}^{a/2} \rho \, x^2 \, dx \, dy +
\]
4. The Hamiltonian of a uniformly magnetized sphere with angular momentum \( \mathbf{L} \) and moment of inertia \( I \) in a magnetic field \( \mathbf{B} \) is

\[
H = \frac{L^2}{2I} + \gamma \mathbf{B} \cdot \mathbf{L},
\]

where \( \gamma \) is the gyromagnetic ratio. Derive an equation of motion for \( \mathbf{L} \) (or its components) and solve it for the case \( \mathbf{B} = (0, 0, B_0) \).

\[
\frac{dL}{dt} = [\mathbf{L}, H]
\]

\[
\frac{d \mathbf{L}^2}{dt} = [\mathbf{L}^2, H] = [\mathbf{L}^2, \gamma \mathbf{B} \cdot \mathbf{L}] = \gamma \mathbf{B} \times \mathbf{L}^2
\]

\[
[\mathbf{L}^2, \mathbf{L} \cdot \mathbf{n}] = 2 \mathbf{n}^2 \times \mathbf{n}^2
\]

\[
[L_i, L_j] = \delta_{ij} L_k
\]
\[
\frac{d \mathbf{L}}{d \tau} = \mathbf{J} \mathbf{B} \times \mathbf{L}
\]

\[L_z = \text{const} = L_z(0)\]

\[L_x(t) = L_z \cos (\mathbf{J} \mathbf{B} \tau + \varphi)\]

\[L_y(t) = L_z \sin (\mathbf{J} \mathbf{B} \tau + \varphi)\]
5. A 1d particle of mass \( m \) moves in a uniform force field, \( V(q) = -Fq \). Find a generating function \( F_1(q, Q) \) of the canonical transformation from \( p(t), q(t) \) to \( P(t) = p(t + \tau), Q(t) = q(t + \tau) \), where \( \tau = \text{const.} \)

\[
\frac{\partial F_1}{\partial \dot{q}} = p, \quad \frac{\partial F_1}{\partial \dot{Q}} = -p
\]

\[
Q(t) = q(t + \tau) = q(t) + \frac{p(t)}{m} \tau + \frac{F \tau^2}{2m}
\]

\[
P(t) = p(t + \tau) = p(t) + F \tau
\]

\[
\overline{Q} = q + \frac{p}{m} \tau + \frac{F \tau^2}{2m}, \quad \overline{P} = p + F \tau
\]

\[
p = \frac{m}{\tau} (Q - \overline{Q}) - \frac{F \tau}{2}, \quad \overline{P} = \frac{m}{\tau} (Q - \overline{Q}) + \frac{F \tau}{2}
\]
\[
\frac{\partial F_1}{\partial \theta} = \rho \\
\frac{\partial F_1}{\partial \varphi} = -\rho
\]
\[
\frac{\partial F_1}{\partial \theta} = \frac{\mu}{c} (\theta - \bar{\theta}) - \frac{F \tau}{2} \\
\frac{\partial F_1}{\partial \varphi} = \frac{\mu}{c} (\theta - \bar{\theta}) + \frac{F \tau}{2}
\]
\[
F_1 = -\frac{\mu}{2c} (\theta - \bar{\theta})^2 - \frac{F \tau}{2} \theta + f(\theta)
\]
\[
-\frac{df}{d\varphi} = \frac{F \tau}{2} \\
f(\theta) = -\frac{F \tau}{2} \theta
\]
\[
F_1(\theta, \varphi) = -\frac{\mu}{2c} (\theta - \bar{\theta})^2 - \frac{F \tau}{2} (\theta + \varphi)
\]