Homework2

1.14 Consider the electrostatic Green functions of Section 1.10 for Dirichlet and Neumann boundary conditions on the surface S bounding the volume V. Apply Green's theorem (1.35) with integration variable $\vec{y}$ and $\phi = G(\vec{x},\vec{y})$, $\psi = G(\vec{x}\,',\vec{y})$, with $\nabla_y^2 G(\vec{z},\vec{y}) = -4\pi\delta(\vec{y}-\vec{z})$. Find an expression for the difference $[G(\vec{x},\vec{x}\,') - G(\vec{x}\,',\vec{x})]$ in terms of an integral over the boundary surface S.

Using ϕ and ψ as indicated in Green's theorem, we have

$$\int_V \left(G(\vec{x},\vec{y})\nabla_y^2 G(\vec{x}\,',\vec{y}) - G(\vec{x}\,',\vec{y})\nabla_y^2 G(\vec{x},\vec{y})\right) d^3y = \oint_S \left(G(\vec{x},\vec{y})\frac{\partial G(\vec{x}\,',\vec{y})}{\partial n_y} - G(\vec{x}\,',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n_y}\right) da_y$$

Since $\nabla_y^2 G(\vec{x},\vec{y}) = -4\pi\delta(\vec{x}-\vec{y})$, the left hand side integrates to $-4\pi[G(\vec{x},\vec{x}\,') - G(\vec{x}\,',\vec{x})]$. Dividing both sides by -4π finally gives

$$G(\vec{x},\vec{x}\,') - G(\vec{x}\,',\vec{x}) = \frac{1}{4\pi}\oint_S \left(G(\vec{x}\,',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n_y} - G(\vec{x},\vec{y})\frac{\partial G(\vec{x}\,',\vec{y})}{\partial n_y}\right) da_y \quad (2)$$

a) For Dirichlet boundary conditions on the potential and the associated boundary condition on the Green function, show that $G_D(\vec{x},\vec{x}\,')$ must be symmetric in $\vec{x}$ and $\vec{x}\,'$.

For the Dirichlet Green's function, $G_D(\vec{x},\vec{y}) = 0$ for $\vec{y}$ on the boundary S. This means that the right hand side of (2) vanishes. Then we automatically find

$$G_D(\vec{x},\vec{x}\,') = G_D(\vec{x}\,',\vec{x})$$

b) For Neumann boundary conditions, use the boundary condition (1.45) for $G_N(\vec{x},\vec{x}\,')$ to show that $G_N(\vec{x},\vec{x}\,')$ is not symmetric in general, but that $G_N(\vec{x},\vec{x}\,') - F(\vec{x})$ is symmetric in $\vec{x}$ and $\vec{x}\,'$, where

$$F(\vec{x}) = \frac{1}{S}\oint_S G_N(\vec{x},\vec{y})\, da_y$$

We use the Neumann boundary condition

$$\frac{\partial G_N(\vec{x},\vec{y})}{\partial n_y} = -\frac{4\pi}{S}$$

for $\vec{y}$ on the boundary S. This means the right hand side of (2) becomes

$$RHS = \frac{1}{S}\oint_S \left(G_N(\vec{x},\vec{y}) - G_N(\vec{x}\,',\vec{y})\right) da_y = F(\vec{x}) - F(\vec{x}\,')$$

where we used the definition of $F(\vec{x})$ given in the problem. This yields

$$G_N^{\text{new}}(\vec{x},\vec{x}\,') \equiv G_N(\vec{x},\vec{x}\,') - F(\vec{x}) = G_N(\vec{x}\,',\vec{x}) - F(\vec{x}\,') \quad (3)$$

which demonstrates that $G_N^{\text{new}}(\vec{x},\vec{x}\,')$ is symmetric in $\vec{x}$ and $\vec{x}'$.

c) Show that the addition of $F(\vec{x}\,)$ to the Green function does not affect the potential $\Phi(\vec{x}\,)$. See problem 3.26 for an example of the Neumann Green function.

What we need to do is to show that the Neumann Green's function solution

$$\Phi(\vec{x}\,) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}\,')G_N(\vec{x},\vec{x}\,')\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}G_N(\vec{x},\vec{x}\,')\,da' \quad (4)$$

is unchanged when we replace G_N by $G_N^{\rm new}$. If we let $\Phi^{\rm new}$ denote the computation using $G_N^{\rm new}$, then

$$\begin{aligned}\Phi^{\rm new}(\vec{x}\,) &= \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}\,')G_N^{\rm new}(\vec{x},\vec{x}\,')\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}G_N^{\rm new}(\vec{x},\vec{x}\,')\,da' \\ &= \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}\,')G_N(\vec{x},\vec{x}\,')\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}G_N(\vec{x},\vec{x}\,')\,da' \\ &\qquad - \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}\,')F(\vec{x}\,)\,d^3x' - \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}F(\vec{x}\,)\,da' \\ &= \Phi(\vec{x}\,) - \frac{F(\vec{x}\,)}{4\pi\epsilon_0}\left(\int_V \rho(\vec{x}\,')\,d^3x' + \epsilon_0\oint_S \frac{\partial\Phi(\vec{x}\,')}{\partial n'}\,da'\right) \\ &= \Phi(\vec{x}\,) - \frac{F(\vec{x}\,)}{4\pi\epsilon_0}\left(q_{\rm enc} - \epsilon_0\oint_S \vec{E}(\vec{x}\,')\cdot\hat{n}'\,da'\right)\end{aligned}$$

where we used the fact that $\hat{n}\cdot\vec{E} = -\hat{n}\cdot\vec{\nabla}\Phi = -\partial\Phi/\partial n$. A simple application of Gauss' law then demonstrates that $\Phi^{\rm new} = \Phi$. Hence we have shown that the addition of $F(\vec{x}\,)$ leaves the solution unchanged. This demonstrates that we can always make G_N symmetric by appropriate modification with F.

Note that from (3) we could instead have defined the symmetric combination $G_N^{\rm bad}(\vec{x},\vec{x}\,') = G_N(\vec{x},\vec{x}\,') + F(\vec{x}\,')$. However this is a bad thing to do, as substitution of $G_N^{\rm bad}$ into (4) will generate an incorrect solution for Φ.

Problem 2.1

A charge q is located at $\mathbf{x}' = d\hat{\mathbf{x}}$, at a distance d from a conducting surface formed by the $x = 0$ plane. In the volume of interest, $x > 0$, the field is that of the original charge and a charge $-q$ at location $\mathbf{x}_1 = -d\hat{\mathbf{x}}$.

a): In the yz-plane it is $\mathbf{E} = E_x\hat{\mathbf{x}} = -\frac{2qd}{4\pi\epsilon_0\sqrt{d^2+\rho^2}^3}\hat{\mathbf{x}}$ with $\rho = \sqrt{x^2+y^2}$. Since $\hat{\mathbf{x}}$ coincides with the normal vector of the conducting surface, the charge density is

$\boxed{\sigma(\rho) = \epsilon_0 E_x = -\frac{qd}{2\pi\sqrt{d^2+\rho^2}^3}}$.

b): The force is attractive, and is $\boxed{\mathbf{F} = -\frac{q^2}{16\pi\epsilon_0 d^2}\hat{\mathbf{x}}}$.

c): The electrostatic pressure is $\frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}}$, where the normal vector of the surface $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. Thus,

$$\mathbf{F} = \hat{\mathbf{x}}\frac{q^2d^2}{8\pi^2\epsilon_0}\int_0^\infty \frac{2\pi\rho\, d\rho}{(d^2+\rho^2)^3} = \hat{\mathbf{x}}\frac{q^2d^2}{4\pi^2\epsilon_0}\left[-\frac{1}{4(d^2+\rho^2)^2}\right]_0^\infty = \hat{\mathbf{x}}\frac{q^2}{16\pi\epsilon_0 d^2}\,, \qquad (10)$$

which is the negative of the result of b) (as expected).

d): The work to be done to move the charge from its location to infinity is

$$W = \int_d^\infty F(d)dd = \frac{q^2}{16\pi^2\epsilon_0}\int_0^\infty \frac{1}{d^2}dd = \left[-\frac{q^2}{16\pi^2\epsilon_0 d}\right]_d^\infty = \frac{q^2}{16\pi^2\epsilon_0 d} > 0 \quad . \qquad (11)$$

e): The potential energy between charge and image charge is $\boxed{W_{\text{pot}} = -\frac{q^2}{8\pi^2\epsilon_0 d}}$, which is **not** equal to $-W$, as one might naively expect, but equal to $-2W$. The customary interpretation of the factor 2 is that in the image problem the field fills all space and is symmetric about the $x = 0$ plane, while in the real problem the field only fills the half-space $x > 0$. Therefore, the potential energy in the image problem can be expected to be twice that of the image problem. Generally, calculations of electrostatic energy are **not** directly transferrable from image to real problems, because the motion of real charges usually implies a motion of the respective image charge(s). The latter motion matters in the electrostatic energy of the image problem, while it does not in the electrostatic energy of the real problem. In contrast, potentials, forces and fields in the volume of interest are same for the real and the image problem.

f): For $q = -e$ and $d = 10^{-10}m$ it is $\boxed{W = 3.6eV}$. This energy is substantial. Note that it corresponds to the work function of typical metals.

Problem 2.2

We consider the problem of a charge q inside a grounded, thin conducting shell with radius a. The calculation of the size and the location of the image charge is analogous to the case of a charge outside a grounded conducting sphere covered in the textbook (swap primed and unprimed variables). For a distance $y < a$ of the charge from the center of the sphere, the image charge $q' = -q\frac{a}{y}$ is located at a distance $y' = y\frac{a^2}{y} > a$. The angle between the vector from the center of the sphere to the observation point and the vector from the center of the sphere to either charge is denoted γ.

a): The potential at a location $\mathbf{x}$ characterized by a radial coordinate x and angle γ is

$$\Phi(x,\gamma) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2-2xy\cos\gamma}} - \frac{a}{y\sqrt{x^2+\frac{a^4}{y^2}-2x\frac{a^2}{y}\cos\gamma}}\right) \tag{12}$$

b): The charge density is the negative of Eq. 2.5 in the textbook,

$$\sigma(\cos\gamma) = \epsilon_0\frac{\partial}{\partial x}\Phi(x,\gamma)|_{x=a} = \frac{q}{4\pi a^2}\frac{a}{y}\frac{1-\frac{a^2}{y^2}}{\sqrt{1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\gamma}^3} \tag{13}$$

c): The force is radially outward and given by

$$\mathbf{F} = \hat{\mathbf{y}}\frac{q^2}{4\pi\epsilon_0}\frac{a}{y}\frac{1}{(y-\frac{a^2}{y})^2} = \hat{\mathbf{y}}\frac{q^2}{4\pi\epsilon_0}\frac{ay}{(a^2-y^2)^2} \tag{14}$$

d): In the following, we identify quantities obtained for the case of a grounded sphere with a subscript I.

Sphere on potential V: Add all charge densities and the potentials of the solutions of the following problems: I=grounded sphere with charge q inside. II=sphere on potential V and no charge inside. The solution of case II is a constant potential V in the shell and its entire interior, because conductors with charge-free internal cavities are equipotential volumes. Outside the shell, the potential of case II drops as $\frac{1}{x}$. The sum of the charge densities of case I and II, and the sum of the potentials, $\Phi = \Phi_I + \Phi_{II}$, satisfy the boundary conditions. That is: the sums produce the correct internal charge distribution and the correct potential on the boundary, respectively. Also, due to the superposition principle, the sum potential and the sum charge distribution are a solution of the Poisson equation. Due to the uniqueness theorem, this must be the only solution for the given surface potential V and the given charge distribution in the volume of interest.

In the conductor and its interior cavity the potential is $\Phi = \Phi_I + \Phi_{II}$, i.e. $\boxed{\Phi = \Phi_I + V}$. The charge density induced on the inner surface is $\boxed{\sigma = \sigma_I}$, since in case II there are no surface charges on the inner surface at all. The internal electric fields derived from Φ and Φ_I are the same, and thus the forces on the charge q are the same, $\boxed{\mathbf{F} = \mathbf{F}_I}$.

Sphere with total charge Q: Again, we think of two solutions and form their sum. Case I is as before. In case I, the total surface charge induced on the inner surface of the shell is $-q$ (think of a Gaussian surface between inner and outer surface of the conducting shell). Also, in case I there is no charge on the outer surface of the shell, because the shell is grounded and, lacking any further information, the exterior potential must be assumed to be zero as well. Case II is a shell with a total charge $Q' = Q + q$ and no charge inside. The charge Q' evenly distributes on the outer surface of the shell; there is no surface charge on the inner surface of the shell. The potential of case II is $V = \frac{Q+q}{4\pi\epsilon_0 a}$.

The solution for the case of a shell with total charge Q and internal charge q is given by summing the potential and the charge distributions of case I and II. Inside the shell it is $\boxed{\Phi = \Phi_I + V = \Phi_I + \frac{Q+q}{4\pi\epsilon_0 a}}$. The charge density induced on the inner surface is $\boxed{\sigma = \sigma_I}$, since in case II there are no surface charges in the inner surface. (The problem doesn't ask for the charge density on the outer surface; it would be $\sigma_{\text{outer}} = \frac{Q+q}{4\pi a^2}$.) The forces on the charge q are the same, $\boxed{\mathbf{F} = \mathbf{F}_I}$.