

Homework1

- 1.4 Each of three charged spheres of radius a , one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as r^n ($n > -3$), has a total charge Q . Use Gauss' theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with $n = -2, +2$.

Because of spherical symmetry, this may be solved by a straightforward application of Gauss' law. In all cases, the electric field (as a function of r) is given by

$$\vec{E} = k \frac{q_{\text{enc}}}{r^2} \hat{r} \quad k = \frac{1}{4\pi\epsilon_0}$$

- i) For the conducting sphere, the charge Q resides on the surface of the sphere (the electric field vanishes inside). Hence

$$\vec{E} = \begin{cases} 0 & r < a \\ k \frac{Q}{r^2} \hat{r} & r > a \end{cases}$$

- ii) For the sphere with uniform charge density, we note that the charge enclosed inside a radius $r < a$ must be proportional to the volume $\frac{4}{3}\pi r^3$. Hence $q_{\text{enc}} = Q(r/a)^3$ and we are left with

$$\vec{E} = \begin{cases} k \frac{Q}{a^3} \vec{r} & r < a \\ k \frac{Q}{r^2} \hat{r} & r > a \end{cases}$$

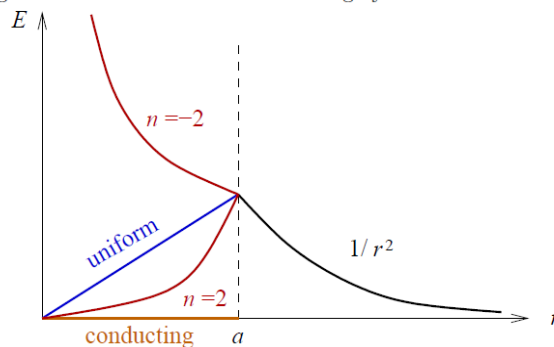
Note that the electric field is linearly proportional to \vec{r} inside the sphere.

- iii) For the sphere with varying charge density $\rho \sim r^n$ the charge enclosed is now proportional to r^{n+3} . Hence $q_{\text{enc}} = Q(r/a)^{n+3}$ and the electric field becomes

$$\vec{E} = \begin{cases} k \frac{Q}{a^3} \left(\frac{r}{a}\right)^n \vec{r} & r < a \\ k \frac{Q}{r^2} \hat{r} & r > a \end{cases}$$

This reduces to the previous case for $n = 0$. Note that the expression for q_{enc} breaks down for $n < -3$. Furthermore, for $n = -3$, $q_{\text{enc}} = Q$ is constant independent of radius r , signifying the charge is concentrated at $r = 0$. This accounts for the point-charge like behavior when $n = -3$. Furthermore, note that in all three cases, the field outside the sphere is identical.

The magnitude of the electric field looks roughly as follows



1.5 The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0} \frac{e^{-\alpha r}}{r} \left(1 + \frac{\alpha r}{2}\right)$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

We may obtain the charge distribution by computing $\rho = -\epsilon_0 \nabla^2 \Phi$. However, since Φ blows up as $r \rightarrow 0$, we must be a bit careful. We first consider $r > 0$

$$\begin{aligned} \rho = -\epsilon_0 \nabla^2 \Phi &= -\frac{q}{4\pi} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \left[e^{-\alpha r} \left(\frac{1}{r} + \frac{\alpha}{2} \right) \right] \\ &= \frac{q}{4\pi} \frac{1}{r^2} \frac{\partial}{\partial r} \left[e^{-\alpha r} \left(1 + \alpha r + \frac{\alpha^2 r^2}{2} \right) \right] = -\frac{q\alpha^3}{8\pi} e^{-\alpha r} \end{aligned}$$

For $r \approx 0$, on the other hand, we may expand

$$\Phi = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{\alpha}{2} + \dots \right) \approx \frac{q}{4\pi\epsilon_0 r}$$

This is the potential of a point charge q at the origin. Hence the complete charge distribution can be written as

$$\rho = q\delta^3(r) - \frac{q\alpha^3}{8\pi} e^{-\alpha r}$$

The first term corresponds to the proton charge, and the second to the negatively charged electron cloud in the $1s$ orbital around the proton.

We can additionally verify that the hydrogen atom is indeed neutral

$$Q = \int \rho d^3x = q - \frac{q\alpha^3}{8\pi} \int_0^\infty e^{-\alpha r} 4\pi r^2 dr = q - \frac{q}{2} \Gamma(3) = 0$$

Problem 1.6

(a) Applying Gauss's law to the plate with a Gaussian pillbox, one gets the electric field due to a flat surface charge distribution to be $\sigma/2\epsilon_0$, where $\sigma = Q/A$ is the surface charge density. The contributions from the two plates add up in between the two plates and cancel outside. The total electric field in between the two plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}.$$

The potential difference between the two plates

$$V \equiv \Phi_+ - \Phi_- = - \int_-^+ \vec{E} \cdot d\vec{\ell} = \frac{Qd}{\epsilon_0 A}$$

The capacitance

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}.$$

(b) Assuming the inner sphere has charge Q and the outer has $-Q$, applying Gauss's law with a Gaussian sphere of radius r ($a < r < b$):

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^3} \vec{r}$$

The potential difference

$$V \equiv \Phi_+ - \Phi_- = - \int_-^+ \vec{E} \cdot d\vec{\ell} = \frac{Q}{4\pi\epsilon_0} \frac{b-a}{ab}$$

The capacitance is therefore

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{b-a}.$$

(c) Again assuming the inner cylinder has charge Q and the outer has $-Q$, applying Gauss's law with a cylindrical surface as the Gaussian surface:

$$\vec{E} = \frac{Q}{2\pi\epsilon_0 r^2} \vec{r}$$

The potential difference

$$V = - \int_-^+ \vec{E} \cdot d\vec{\ell} = \frac{Q}{2\pi\epsilon_0} \ln \frac{b}{a}$$

The capacitance per unit length

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0}{\ln \frac{b}{a}}$$

Problem 1.9(a) *Parallel plate capacitor*

The negatively charged plate experiences a field of

$$E = \frac{\sigma}{2\epsilon_0} = \frac{Q}{2\epsilon_0 A}$$

due to the positively charged plate, where Q is the total charge on the plate. Therefore, the attractive force between the two plates is

$$F = QE = \frac{Q^2}{2\epsilon_0 A}$$

Parallel cylinder capacitor

Again, one conductor experiences an electric field of

$$E \approx \frac{Q}{2\pi\epsilon_0 d}$$

from the other conductor. Here Q is the charge per unit length. Therefore, the attractive force per unit length between them is

$$F = QE = \frac{Q^2}{2\pi\epsilon_0 d}$$

(b) The force should be the same except that Q should be replaced by CV .*Parallel plate capacitor*

$$F = \frac{\epsilon_0 AV^2}{2d^2}$$

Parallel cylinder capacitor

$$F = \frac{\pi\epsilon_0 V^2}{2d \ln(b/a)^2}$$