Physics 503
Midterm Exam 2011

Problem 1 Within a sphere of radius R there is a non-zero charge density,

$$\rho(r,\theta,\varphi) = \rho_0 \frac{R}{r} \sin^2\theta. \tag{1}$$

The sphere is surrounded by an infinite vacuum.

a) Determine the total charge.
b) Determine the potential $\Phi(\mathbf{r})$ outside the sphere.
c) Determine the potential $\Phi(\mathbf{r})$ inside the sphere.

Solution 1 The potential can be obtained by

$$\Phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} \tag{2}$$

The result can be expressed in terms of Legendre polynomials using the expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi') \tag{3}$$

We get

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty} P_l(\cos\theta)2\pi\int_{-1}^{1} d(cos\theta')P_l(\cos\theta')\int_0^R dr' r'^2 \frac{r_<^l}{r_>^{l+1}}\rho(r',\theta') \tag{4}$$

where we used the relation

$$Y_{l,0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \tag{5}$$

Inserting ρ gives

$$\Phi(\mathbf{r}) = \frac{\rho_0 R}{2\epsilon_0}\sum_{l=0}^{\infty} P_l(\cos\theta)\int_{-1}^{1} dx(1-x^2)P_l(x)\int_0^R dr' r' \frac{r_<^l}{r_>^{l+1}} \tag{6}$$

The integral over $\cos\theta$ gives

$$\int_{-1}^{1} dx P_l(x)(1-x^2) = \frac{4}{3}\delta_{l=0} - \frac{4}{15}\delta_{l=2} \tag{7}$$

Integral over radius is

$$\int_0^R dr' \frac{r_<^l}{r_>^{l+1}} = \begin{cases} \frac{1}{r^{l+1}}\int_0^R dr' r'^{l+1} & r > R \\ \frac{1}{r^{l+1}}\int_0^r dr' r'^{l+1} + r^l \int_r^R dr' \frac{1}{r'^l} & r < R \end{cases} \tag{8}$$

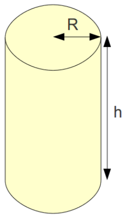
which becomes

$$\int_0^R dr' \frac{r_<^l}{r_>^{l+1}} = \begin{cases} \frac{1}{l+2}\frac{R^{l+2}}{r^{l+1}} & r > R \\ \frac{1}{l+2}r - \frac{1}{l-1}(\frac{r^l}{R^{l-1}} - r) & r < R \end{cases} \tag{9}$$

Inserting integrals into expression for potential gives

$$\Phi(\mathbf{r}) = \begin{cases} \frac{\rho_0 R^2}{3\epsilon_0}\left[\frac{R}{r} - \frac{1}{10}\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right] & r > R \\ \frac{\rho_0 R^2}{3\epsilon_0}\left[(2-\frac{r}{R}) - \frac{r}{R}(\frac{1}{2} - \frac{2}{5}\frac{r}{R})P_2(\cos\theta)\right] & r < R \end{cases} \tag{10}$$

Problem 2 a) Find the Dirichlet Green's function for a conducting cylinder with radius R and height h, which is grounded (potential on all sites vanishes). Express it in terms of a series of the solutions of Laplace Eq. in cylindrical coordinates.



Note: You can use the following representation of the delta function

$$\delta(z-z') = \frac{2}{h}\sum_{n=1}^{\infty} \sin(n\pi\frac{z}{h})\sin(n\pi\frac{z'}{h}) \tag{11}$$

Also note that the modified Bessel functions satisfy the following relations

$$K_m(x)\frac{dI_m(x)}{dx} - I_m(x)\frac{dK_m(x)}{dx} = \frac{1}{x} \tag{12}$$

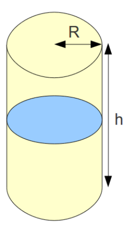
$$I_m(x \ll 1) \approx \frac{1}{m!}\left(\frac{x}{2}\right)^m \tag{13}$$

$$K_m(x \ll 1) \approx \frac{(m-1)!}{2}\left(\frac{2}{x}\right)^m \qquad \text{if } m > 0 \tag{14}$$

$$K_0(x \ll 1) \approx -\ln(x/2) \tag{15}$$

$$\tag{16}$$

b) At the half-height of the cylinder a thin charged disc is inserted. The charge distribution σ is constant on the disc. What is potential outside and inside the conducting cylinder? You can use the above derived Green's function to find the potential.



Note: You might find the following relations useful

$$\int dx\, I_0(x)\, x = x\, I_1(x) \tag{17}$$

$$\int dx\, K_0(x)\, x = -x\, K_1(x) \tag{18}$$

c) Find the electric field at the bottom, at the top and in the middle of the cylinder close to the axis of the cylinder ($r << R$). What is the relation between electric field at the top and at the bottom?

Solution 2 We will determine the Green's function in ab-initio way. The Green's function satisfies

$$\nabla^2_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{19}$$

We write the delta function in cylindrical coordinates and express it in terms of eigenfunctions satisfying the boundary condition in variables ϕ and z

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{\delta(r-r')}{r'} \quad \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \quad \frac{2}{h}\sum_{n=1}^{\infty}\sin(n\pi\frac{z}{h})\sin(n\pi\frac{z'}{h}) \tag{20}$$

We search for the Green's function using the following ansatz

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi}\frac{2}{h}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin(n\pi\frac{z}{h})\sin(n\pi\frac{z'}{h}) g_{nm}(r,r') \tag{21}$$

Applying ∇^2 on ansatz for G and using definition for the Green's function, we conclude

$$\frac{1}{r'}\frac{\partial}{\partial r'}(r'\frac{\partial g_{nm}}{\partial r'}) - \left(\frac{m^2}{r'^2} + k_n^2\right) g_{nm} = -4\pi\frac{\delta(r-r')}{r'} \tag{22}$$

Here we used $k_n \equiv \frac{n\pi}{h}$. Clearly g_{nm} satisfies bessel dif. equation in all points except $r = r'$. The solutions of this dif. eq. are modified bessel functions $I_m(k_n r)$ and $K_m(k_n r)$. If $\psi_1(r')$ is the solution for $r' < r$ and $\psi_2(r')$ is solution for $r > r'$, we have a general symmetric solution of the type

$$g_{nm}(r,r') = \psi_1(k_n r_<)\psi_2(k_n r_>) \tag{23}$$

The delta function in the diff. equation constrains the two functions

$$\left[\frac{\partial g_{nm}}{\partial r'}\right]_{r'=r-\epsilon}^{r'=r+\epsilon} = -\frac{4\pi}{r'} \tag{24}$$

or

$$\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx} = \frac{4\pi}{x} \tag{25}$$

We need to choose ψ_1 and ψ_2 such that g_{nm} is finite at $r = 0$ and vanishes at $r = R$. The correct choice is

$$\psi_1(k_n r) = \mathcal{A}\, I_m(k_n r) \tag{26}$$

$$\psi_2(k_n r) = K_m(k_n r) - I_m(k_n r)\frac{K_m(k_n R)}{I_m(k_n R)} \tag{27}$$

where $\mathcal{A}$ is a constant to be determined. Inserting into Eq.25, we get

$$\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx} = \mathcal{A}(K_m - cI_m)I_m' - \mathcal{A}I_m(K_m' - cI_m') = \mathcal{A}(K_m I_m' - I_m K_m') = \frac{\mathcal{A}}{x} \tag{28}$$

which determines $\mathcal{A} = 4\pi$. We here used $c \equiv \frac{K_m(k_n R)}{I_m(k_n R)}$.

The Green's function therefore is

$$G(\mathbf{r},\mathbf{r}') = \frac{4}{h}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin(n\pi\frac{z}{h})\sin(n\pi\frac{z'}{h})I_m(k_n r_<)\left[K_m(k_n r_>) - I_m(k_n r_>)\frac{K_m(k_n R)}{I_m(k_n R)}\right] \tag{29}$$

where $k_n = n\pi/h$.

The disk of constant charge can be expressed in terms of volume charge by

$$\rho(\mathbf{r}) = \sigma\delta(z - \frac{h}{2}) \tag{30}$$

using the Green's function method we determine Φ by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}') \tag{31}$$

Using the above derived Green's function we get

$$\Phi(\mathbf{r}) = \frac{2\sigma}{\epsilon_0 h}\sum_{n=1}^{\infty}\sin(n\pi\frac{z}{h})\sin(\frac{n\pi}{2})\int_0^R dr' r' I_0(k_n r_<)\left[K_0(k_n r_>) - I_0(k_n r_>)\frac{K_0(k_n R)}{I_0(k_n R)}\right] \tag{32}$$

The integration has to be split into $r' < r$ and $r' > r$, which gives

$$\int_0^R dr' r' I_0(k_n r_<)\left[K_0(k_n r_>) - I_0(k_n r_>)\frac{K_0(k_n R)}{I_0(k_n R)}\right] = \tag{33}$$

$$\frac{1}{k_n^2}\left\{(k_n r)\left[I_0(k_n r)K_1(k_n r) + I_1(k_n r)K_0(k_n r)\right] - I_0(k_n r)(k_n R)\left[K_1(k_n R) + I_1(k_n R)\frac{K_0(k_n R)}{I_0(k_n R)}\right]\right\} \tag{34}$$

We introduce a number

$$\mathcal{D}_n \equiv (k_n R)\left[K_1(k_n R) + I_1(k_n R)\frac{K_0(k_n R)}{I_0(k_n R)}\right]$$

to simplify the potential

$$\Phi(\mathbf{r}) = \frac{2\sigma h}{\epsilon_0 \pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi \frac{z}{h}) \sin(\frac{n\pi}{2})}{n^2} \left\{ (k_n r) \left[I_0(k_n r) K_1(k_n r) + I_1(k_n r) K_0(k_n r) \right] - I_0(k_n r) \mathcal{D}_n \right\} \tag{35}$$

In the limit of $r \ll R$, we have

$$\lim_{r \to 0} \left\{ (k_n r) \left[I_0(k_n r) K_1(k_n r) + I_1(k_n r) K_0(k_n r) \right] - I_0(k_n r) \mathcal{D}_n \right\} = 1 - \mathcal{D}_n \tag{36}$$

and therefore

$$\Phi(r = 0, z) = \frac{2\sigma h}{\epsilon_0 \pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi \frac{z}{h}) \sin(\frac{n\pi}{2})}{n^2} (1 - \mathcal{D}_n) \tag{37}$$

The electric field is

$$E_z(r = 0, z) = -\frac{2\sigma}{\epsilon_0 \pi} \sum_{n=1}^{\infty} \frac{\cos(n\pi \frac{z}{h}) \sin(\frac{n\pi}{2})}{n} (1 - \mathcal{D}_n) \tag{38}$$

In the middle, we have combination of $\cos(\frac{n\pi}{2}) \sin(\frac{n\pi}{2})$, which vanishes for any integer n. The electric field in the middle thus vanishes.

At the bottom we have

$$E_z(r = 0, z = 0) = -\frac{2\sigma}{\epsilon_0 \pi} \sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n} (1 - \mathcal{D}_n) = -\frac{2\sigma}{\epsilon_0 \pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (1 - \mathcal{D}_{2n+1}) \tag{39}$$

and at the top we have

$$E_z(r = 0, z = 0) = -\frac{2\sigma}{\epsilon_0 \pi} \sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2}) \cos(n\pi)}{n} (1 - \mathcal{D}_n) = \frac{2\sigma}{\epsilon_0 \pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (1 - \mathcal{D}_{2n+1}) \tag{40}$$

The electric field thus changes sign from top to bottom.