Physics 502: Problem Set 4 (DUE TUESDAY 3/7)

1) Shankar 19.3.1 p. 533. Note that, as claimed near the top of p. 532, this diverges in the
limit of a Coulomb interaction ($r_0 \rightarrow \infty$).

2) Shankar 19.3.3 p. 533 (Gaussian potential).

3) Sakurai Chapter 7 Problem 3 (p. 441-442) (finite spherical barrier/well).

4) Let me briefly recap the result derived in class about inelastic scattering in the first
Born approximation. The Hamiltonian is $H = H_{\text{probe}} + H_{\text{targ}} + V$. The probe particle is
a non-relativistic particle of charge $Q$ and mass $M$, $H_{\text{probe}} = P^2/2M$, making a transition
from $|K_i\rangle$ to $|K_f\rangle$ having energy $\hbar^2 K_i^2/2M$ and $\hbar^2 K_f^2/2M$ respectively. The target system
has unperturbed Hamiltonian $H_{\text{targ}} = \sum_j p_j^2/2m + V(r_1, r_2, ...)$, and we define the density
operator $\rho(r) = \sum_j q_j \delta(r - r_j)$ and its Fourier transform

$$\tilde{\rho}(K) = \int d^3 r e^{-iK \cdot r} \rho(r) = \sum_j q_j e^{-iK \cdot r_j}$$

Finally, the perturbation is the Coulomb interaction,

$$V = \sum_j \frac{Q q_j}{|r_j - R|} = \int d^3 r \frac{Q \rho(r)}{|r - R|}$$

where $R$ is the probe coordinate. Then

$$\frac{d\sigma}{d\Omega} = \left( \frac{M}{2\pi\hbar^2} \right)^2 K_f K_i \left( \frac{4\pi Q}{K^2} \right)^2 |\tilde{\rho}_{n,s}(K)|^2$$

a) Consider the case that $K = K_f - K_i$ is small, and show that the leading approximation
for small $K$

$$\frac{d\sigma}{d\Omega} = \left( \frac{M}{2\pi\hbar^2} \right)^2 K_f K_i \left( \frac{4\pi Q}{K^2} \right)^2 |K \cdot d_{n,s}|^2$$

where $d$ is the dipole operator $\int d^3 r r \rho(r)$.

b) Returning to arbitrary $K$, consider now the case that no transition actually occurs in
the target system, so that $n = s$. Show that the differential scattering cross section is
what you would expect from elastic first-order Born theory for scattering off a static
potential given by $U(R) = \int d^3 r Q \rho_{ss}(r)/|r - R|$. 

c) Now forget about the special cases of parts (a) and (b). Let's do a case where we can get an exact solution. The target system consists of only a single particle bound to the origin by a 3D simple harmonic oscillator potential, \( H_{\text{targ}} = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 r^2 \); we work here with the \( |n_x n_y n_z \rangle \) basis based on Cartesian separability (not the one based on spherical \( Y_l^m \)s). Obtain the differential scattering cross section \( d\sigma/d\Omega \) for a probe particle which enters from the \(-\hat{x}\) direction with energy \( 3\hbar\omega_0 \) and exits along the \(+\hat{z}\) direction, causing a transition in the target system from ground state \( |000\rangle \) to excited final state \( |100\rangle \) (i.e., adding one quantum of excitation in the \( x \) direction). You may use without proof the following results for certain matrix elements of the one-dimensional SHO: 
\[
\langle 0 | e^{-ikx} | 0 \rangle = e^{-k^2/4\alpha}, \quad \langle 1 | e^{ikx} | 0 \rangle = (ik/\sqrt{2\alpha})e^{-k^2/4\alpha},
\]
where \( \alpha = m\omega_0/\hbar \). For consistency of solutions, please try to eliminate as many variables as possible (e.g., \( K_f \), \( K_i \), etc.) in favor of \( \omega_0 \) in your final answer.