1. Two quantum operators have the matrix representation

\[
A = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

(a) A system is in quantum state \(|\psi\rangle\) that is in an eigenfunction of operator \(B\), corresponding to eigenvalue \(-1\). Then for this state, what are \(\langle B \rangle\) and \(\Delta B\)?

**Ans.:** \(\langle B \rangle = -1\), and \(\Delta B = 0\), because observable is sharply defined in an eigenstate.

(b) First \(B\) is measured and the result is \(b = -1\). What is the state of the system after the measurement?

**Ans.:** It is in the eigenstate of \(B\) with eigenvalue \(-1\), which is \(|b = -1\rangle = \frac{1}{\sqrt{2}} (0, 1, -1)\).

(c) Immediately afterwards, \(A\) is measured. What is the probability to find \(a = 1\)?

There are two states with eigenvalue \(a = 1\). These are

\[
|a_1\rangle = \frac{1}{\sqrt{2}} (i, 1, 0) \\
|a_2\rangle = (0, 0, 1)
\]

The probability is therefore \(P(1) = |\langle a_1|b\rangle|^2 + |\langle a_2|b\rangle|^2 = \frac{3}{4}\)

(d) Assuming that \(a = 1\) was indeed found in (c), what is the state of the system after the measurement of \(A\)?

**Ans.:** We will find the system in the projected state

\[|\psi_{final}\rangle = (|a_1\rangle \langle a_1| + |a_2\rangle \langle a_2|)|b\rangle \propto \frac{1}{\sqrt{6}} (i, 1, -2)\]

2. A particle of mass \(m\) is in 1D potential of the form

\[
V(x) = \begin{cases} 
\infty & \text{for } x < 0 \\
-V_0 & \text{for } 0 \leq x \leq a \\
0 & \text{for } x > a
\end{cases}
\]
Suppose that there are two bound states, one at $E = 0$ and the other deeper down at $E = -E_b < 0$.

(a) Find the potential $V_0$, which allows such solutions.
(b) Sketch the two bound state solutions in this problem.

**Ans.:** The wave function must have the form
\[
\psi(x) = \begin{cases} 
A \sin(kx) & \text{for } 0 < x < a \\
Be^{-\kappa x} & \text{for } x > 1 
\end{cases}
\] (5)

Due to continuity of the solution and its derivative at $a$ we have
\[
A \sin(ka) = Be^{-\kappa a}
\] (6)
\[
Ak \cos(ka) = -\kappa Be^{-\kappa a}
\] (7)

which gives the equation
\[
\tan(ka) = -\frac{k}{\kappa}
\] (8)

Since the solution satisfies the Schroedinger equation, we have
\[
\frac{\hbar^2 k^2}{2m} - V_0 = E
\]
\[
-\frac{\hbar^2 \kappa^2}{2m} = E
\] (9) (10)

or
\[
k^2 + \kappa^2 = \frac{2m}{\hbar} V_0.
\] (11)

For the second solution we must have $E = 0$ and hence $\kappa^2 = 0$, which immediately leaves only a discrete number of solutions, those that satisfy $\tan(ka) = -\infty$ (see Eq. 8). The possibilities are $k_2 a = \pi/2 + n\pi + \delta$, whit $n$ integer and $\delta$ infinitesimal number.

Next we combine Eq. 11 and Eq. 8 which defines the bound state solutions
\[
\tan(ka) = -\frac{ka}{\sqrt{\gamma^2 - (ka)^2}}
\] (12)
with
\[ \gamma = \sqrt{\frac{2mV_0a^2}{\hbar^2}}. \]

By plotting the function on the left and on the right, we see that the first solution would occur between \( k_1a \in (\pi/2, \pi) \), hence the second solution must be \( k_2a = 3\pi/2 \). In this solution \( \kappa_2 = 0 \), hence we immediately determine
\[ \frac{2m}{\hbar}V_0 = \left( \frac{3\pi}{2a} \right)^2 \]
and therefore
\[ V_0 = \left( \frac{3\pi}{2} \right)^2 \frac{\hbar}{2ma^2} \tag{13} \]

3. Consider a 1D problem with a symmetric potential (such as the harmonic oscillator), which has the ground state solution \( |g\rangle \) and the first excited solution \( |e\rangle \), with the corresponding energies \( E_0 \) and \( E_1 \). Let’s say that only these two states are accessible to an electron.

We know that there are two non-interacting identical electrons in the potential, and at one particular time we determined that one was in the state \( |\psi_1\rangle \) and one in the state \( |\psi_2\rangle \) with
\[ |\psi_1\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle) \tag{14} \]
\[ |\psi_2\rangle = \frac{1}{\sqrt{2}}(|g\rangle - |e\rangle). \tag{15} \]

(a) What was the correct wave function for two identical fermions in such a configuration, when we take into account the statistics?

Ans.:
\[ |\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle \otimes |\psi_2\rangle - |\psi_2\rangle \otimes |\psi_1\rangle) \tag{16} \]
(b) Calculate $\langle E \rangle$, $\langle E^2 \rangle$ and $\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}$ for such two electron state.

**Ans.:** For this part, we want to rewrite $|\psi\rangle$ in terms of $|g\rangle$ and $|e\rangle$. Straightforward algebra gives that

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|e\rangle \otimes |g\rangle - |g\rangle \otimes |e\rangle) \quad (17)$$

We realize that this is an eigenstate of $H$, with the eigenvalue $E_0 + E_1$, i.e.,

$$H |\psi\rangle = (E_0 + E_1) |\psi\rangle \quad (18)$$

therefore $\langle E \rangle = E_0 + E_1$ and $\langle E^2 \rangle = (E_0 + E_1)^2$ and $\Delta E = 0$.

(c) Calculate the average position $\langle X \rangle$ where position operator is $\hat{X} = \hat{X}_1 + \hat{X}_2$.

**Ans.:** We need to calculate

$$\langle X \rangle = \frac{1}{2} ((\langle e \rangle \otimes \langle g \rangle - \langle g \rangle \otimes \langle e \rangle)(X_1 + X_2)(|e\rangle \otimes |g\rangle - |g\rangle \otimes |e\rangle)) = 0 \quad (19)$$

which vanishes because $\langle g|X|g\rangle = 0$ and $\langle e|X|e\rangle = 0$ since the potential is symmetric.