1) Consider a system of two non-identical fermions, each with spin $1/2$. One is in a state with $S_1 = +\frac{\hbar}{2}$, while the other is in a state with $S_2 = -\frac{\hbar}{2}$. What is the probability of finding the system in a state with total spin quantum numbers $s = 1$, $m_s = 0$, where $m_s$ refers to the $z$-component of the total spin?

   a) First, find the eigenstate of the operator $S_1$ with the eigenvalue $\frac{\hbar}{2}$. Also find the eigenstate of $S_2$ with the eigenvalue $-\frac{\hbar}{2}$.

   **Answ.** The eigenvectors are
   
   $|S_1 = +\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$

   $|S_2 = -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - i|\downarrow\rangle)$

   b) Using the rules for summation of angular momenta, find the expression for the state $|s = 1, m_s = 0\rangle$.

   **Answ.** The triplet state is
   
   $|s = 1, m_s = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

   c) Calculate the probability.

   **Answ.** Since the fermions are not identical, the wave function of the system is the product wave function of $|S_1 = +\frac{1}{2}\rangle$ and $|S_2 = -\frac{1}{2}\rangle$, i.e.,
   
   $|\psi\rangle = \frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle - i|\downarrow\rangle)$

   The probability is thus
   
   $P = |\langle s = 1, m_s = 0 | \psi \rangle|^2 = \frac{1}{2\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle - i|\uparrow\downarrow\rangle - i|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle)^2 = \frac{1}{2\sqrt{2}} = \frac{1}{4}$

2) Consider two spin-1 particles that occupy the state $|s_1 = 1, m_1 = 1; s_2 = 1, m_2 = 0\rangle$. 
What is the probability of finding the system in an eigenstate of the total spin \( S^2 \) with quantum number \( s = 1 \)? What is the probability for \( s = 2 \)?

**Answ.** Using Clebsh-Gordan coefficients for addition of angular momenta, we have

\[
|s = 1, m_s = 1\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle - |0, 1\rangle) \quad (1)
\]

\[
|s = 1, m_s = 0\rangle = \frac{1}{\sqrt{2}}(|1, -1\rangle - |-1, 1\rangle) \quad (2)
\]

\[
|s = 1, m_s = -1\rangle = \frac{1}{\sqrt{2}}(|0, -1\rangle - |-1, 0\rangle) \quad (3)
\]

On the right-hand side the notation is \(|m_1, m_2\rangle\). Similarly, we can obtain

\[
|s = 2, m_s = 2\rangle = |1, 1\rangle \quad (4)
\]

\[
|s = 2, m_s = 1\rangle = \frac{1}{\sqrt{2}}(|0, 1\rangle + |1, 0\rangle) \quad (5)
\]

\[
|s = 2, m_s = 0\rangle = \frac{1}{\sqrt{6}}(|1, -1\rangle + |-1, 1\rangle) + \sqrt{\frac{2}{3}}|0, 0\rangle \quad (6)
\]

\[
|s = 2, m_s = -1\rangle = \frac{1}{\sqrt{2}}(|0, -1\rangle + |-1, 0\rangle) \quad (7)
\]

\[
|s = 2, m_s = -2\rangle = |-1, -1\rangle \quad (8)
\]

The probabilities are then

\[
P(s = 1) = |\langle 1, 0|s = 1, ms = 1\rangle|^2 + |\langle 1, 0|s = 1, ms = 0\rangle|^2 + |\langle 1, 0|s = 1, ms = -1\rangle|^2 = 1/2
\]

\[
P(s = 2) = |\langle 1, 0|s = 2, ms = 1\rangle|^2 + |\langle 1, 0|s = 2, ms = 0\rangle|^2 + |\langle 1, 0|s = 2, ms = -1\rangle|^2 = 1/2
\]

3) a) Construct the spin singlet \((S = 0)\) state and the spin triplet \((S = 1)\) states of a two electron system.

**Answ.** singlet:

\[
|0, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \quad (9)
\]

triplets:

\[
|1, 1\rangle = |\uparrow\uparrow\rangle \quad (10)
\]

\[
|1, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \quad (11)
\]

\[
|1, -1\rangle = |\downarrow\downarrow\rangle \quad (12)
\]

b) In the experiment we have two electrons, which are in the spin-singlet state. They move in the opposite direction along the \(y\)-axis, and two observers \(A\) and \(B\) measure the spin state of each electron. \(A\) measures the spin component along the \(z\) axis, and \(B\) measures the spin component along an axis making an angle
\(\theta\) with the \(z\) axis in the \(xz\)-plane. Suppose that \(A\)'s measurement yields a spin down state and subsequently \(B\) makes a measurement. What is the probability that \(B\)'s measurement yields an up spin (measured along an axis making an angle \(\theta\) with the \(z\)-axis)?

The explicit formula for the representation of the rotation operator \(\exp(-i\mathbf{S} \cdot \hat{n}\theta/\hbar)\) in the spin space is given by the spin \(1/2\) Wigner matrix

\[
D^{(1/2)}(\hat{n}, \theta) = \begin{pmatrix} \cos(\theta/2) - in_z \sin(\theta/2) & (-in_x - n_y) \sin(\theta/2) \\ (-in_x + n_y) \sin(\theta/2) & \cos(\theta/2) + in_z \sin(\theta/2) \end{pmatrix}
\]

and \(\hat{n} = n_x \mathbf{\hat{e}}_x + n_y \mathbf{\hat{e}}_y + n_z \mathbf{\hat{e}}_z\) (\(|\hat{n}| = 1\)) is the axis of rotation.

**Answ.:** Since the state of two electrons is singlet, and we know that the first electron points down, the second has to point up in the same coordinate system. But observer \(B\) is rotated by \(\theta\) around \(y\) axis, hence we need to find how spin-up looks in the rotated coordinate system. We thus apply \(D^{(1/2)}(\hat{e}_y, \theta)\) on \((1,0)\) to get

\[
|\psi_B\rangle = (\cos(\theta/2), \sin(\theta/2))
\]

The probability for up-spin is thus \(P(|\uparrow\rangle) = \cos^2(\theta/2)\) and for down-spin \(P(|\downarrow\rangle) = \sin^2(\theta/2)\).

4) The Wigner-Eckart theorem is given by

\[
\langle n'j'm'|T^{(0)}|njm\rangle = \langle j'm'|lq,jm\rangle \frac{\langle\langle n'j'|T^{(0)}|nj\rangle\rangle}{\sqrt{2j + 1}}
\]

a) Explain the meaning of the two terms on the right hand side.

**Answ.:** The first term is the Clebsch-Gordan coefficient, which encodes the geometric properties of the matrix element under rotation. The second is the reduced matrix element, which is a common coefficient for all \(m,m'\) quantum numbers.

b) The interaction of the electromagnetic field with a charged particle is given by

\[
\Delta H = \frac{e}{2m} \mathbf{A} \cdot \mathbf{p}
\]

If the electromagnetic fields are in the form of a plane wave, then \(\mathbf{A} = A_0 \mathbf{\hat{e}} e^{i\mathbf{k} \cdot \mathbf{r}}\), where \(\mathbf{\hat{e}}\) is the polarization of the plane wave. Assuming that the wavelength \(\lambda = 2\pi/\mathbf{k}\) is much larger than the atomic size, we may write

\[
\mathbf{A} = A_0 \mathbf{\hat{e}} (1 + i\mathbf{k} \cdot \mathbf{r} + \cdots)
\]

such that

\[
\Delta H \approx \frac{e}{2m} A_0 \mathbf{\hat{e}} \cdot \mathbf{p} (1 + i\mathbf{k} \cdot \mathbf{r})
\]

Here we kept both the dipole (the first term), and the quadrupole terms (the second term).
If the field is polarized along the $x$-axis ($\hat{\epsilon} = \hat{e}_x$), and the wave propagation is along the $z$-axis ($\mathbf{k} = k\hat{e}_z$) express the Hamiltonian in terms of spherical harmonics. Note that $\mathbf{p}$ is a vector operator, and transforms under rotation as $\mathbf{r}$. For symmetry consideration you may therefore replace $\mathbf{p}$ by $C\mathbf{r}$.

**Answ.**: The Hamiltonian for the above configuration is

$$ \Delta H = \frac{e}{2m} A_0 C (x + i k x z) \quad (16) $$

Using the expressions for $Y_{lm}'$, we can get

$$ x = \sqrt{\frac{2\pi}{3}} r (Y_{1,-1} - Y_{1,1}) \quad (17) $$

$$ xz = \sqrt{\frac{2\pi}{15}} r^2 (Y_{2,-1} - Y_{2,1}) \quad (18) $$

hence

$$ \Delta H = \frac{e}{2m} A_0 C \sqrt{\frac{2\pi}{3}} r (Y_{1,-1} - Y_{1,1} + i \frac{kr}{\sqrt{5}} (Y_{2,-1} - Y_{2,1})) \quad (19) $$

c) For the above configuration, derive the selection rules for the dipole and the quadrupole transitions, by considering the transition probability matrix elements $|\langle \psi_f | \Delta H | \psi_i \rangle|^2 = |\langle l_f m_f | \Delta H | l_i m_i \rangle|^2$. Note: selection rules state under which conditions is a transition possible.

**Answ.**: The dipole matrix elements are proportional to

$$ |\langle l_f m_f | \Delta H_1 | l_i m_i \rangle|^2 \propto |\langle l_f m_f | Y_{1,-1} - Y_{1,1} | l_i m_i \rangle|^2 \propto |\langle l_f m_f | Y_{11,1} - Y_{11,-1} | l_i m_i \rangle|^2 \propto |\langle l_f m_f | l_i m_i \rangle|^2 |\langle l_f m_f | l_i m_i \rangle|^2 \quad (20) $$

hence $|m_f - m_i| = 1$, and $|l_f - l_i| \leq 1$.

The quadrupole matrix elements are

$$ |\langle l_f m_f | \Delta H_2 | l_i m_i \rangle|^2 \propto |\langle l_f m_f | Y_{2,-1} - Y_{2,1} | l_i m_i \rangle|^2 \propto |\langle l_f m_f | Y_{21,1} - Y_{21,-1} | l_i m_i \rangle|^2 \propto |\langle l_f m_f | l_i m_i \rangle|^2 |\langle l_f m_f | l_i m_i \rangle|^2 \quad (21) $$

hence $|m_f - m_i| = 1$, and $|l_f - l_i| \leq 2$.

The explicit expressions for the spherical harmonics for $l = 1, 2$ are given by

$$ Y_{1,1} = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} x + iy \quad Y_{1,0} = \frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{z}{r} \quad (22) $$

$$ Y_{2,2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \frac{(x + iy)^2}{r^2} \quad Y_{2,1} = -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \frac{(x + iy)z}{r^2} \quad Y_{2,0} = \frac{1}{4} \sqrt{\frac{5}{\pi}} \frac{2z^2 - x^2 - y^2}{r^2} \quad (23) $$

and $Y_{1,-m} = (-1)^m Y_{1,m}^*$. 