1) Using the matrix elements of the operator $L_x$ in the subspace for $l = 1$ derived in the previous homework, show that the matrix for arbitrary rotations around the x-axis is given by

$$D_{mm'}(\theta) = \exp(-i\theta L_x/\hbar) = \begin{pmatrix}
\frac{1}{2} \cos \theta + \frac{i}{2} \sin \theta & -\frac{i}{\sqrt{2}} \sin \theta & \frac{1}{2} \cos \theta - \frac{i}{2} \\
-\frac{i}{\sqrt{2}} \sin \theta & \cos \theta & -\frac{i}{\sqrt{2}} \sin \theta \\
\frac{1}{2} \cos \theta - \frac{i}{2} & -\frac{i}{\sqrt{2}} \sin \theta & \frac{1}{2} \cos \theta + \frac{i}{2}
\end{pmatrix} \quad (1)$$

**Ans.:** One can diagonalize $3 \times 3$ matrix of the operator $L_x$, and derive the matrix of rotation. The alternative derivation reliance on the Taylor series of the exponent. One can notice that

$$
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}^2 = 
\begin{pmatrix}
1 & 0 & 1 \\
0 & 2 & 0 \\
1 & 0 & 1
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}^3 = 
\begin{pmatrix}
0 & 2 & 0 \\
2 & 0 & 2 \\
0 & 2 & 0
\end{pmatrix}
$$

hence the Taylor series

$$D_{mm'}(\theta) = \exp(-i\theta L_x/\hbar) = \exp\left(-i\frac{\theta}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}\right) = \sum_n \frac{1}{n!} \left(-i\frac{\theta}{\sqrt{2}}\right)^n \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}^n \quad (2)$$

gives

$$D_{mm'}(\theta) = 1 + \sum_{n=1,3,...} \frac{1}{n!} \left(-i\frac{\theta}{\sqrt{2}}\right)^n \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} 2^{(n-1)/2} + \sum_{n=2,4,...} \frac{1}{n!} \left(-i\frac{\theta}{\sqrt{2}}\right)^n \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix} 2^{(n-1)/2}$$

$$D_{mm'}(\theta) = 
\begin{pmatrix}
1/2 & 0 & -1/2 \\
0 & 0 & 0 \\
-1/2 & 0 & 1/2
\end{pmatrix} + \frac{1}{\sqrt{2}} (-i \sin \theta) \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} + \frac{1}{2} \cos \theta \begin{pmatrix}
1 & 0 & 1 \\
0 & 2 & 0 \\
1 & 0 & 1
\end{pmatrix} \quad (3)$$
which is equivalent to the given matrix above.

Show that applying this matrix for the case of \( \theta = \pi \) on the eigenfunction \(|l = 1, m = 1\rangle\) gives the same result as rotating explicitly the function \( Y_{1,1}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} \) by 180-degrees around the x-axis.

**Ans.:** The rotation by 180 degrees is

\[
D(\pi) = \begin{pmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\]

hence rotating \((1, 0, 0)\) gives \((0, 0, -1)\).

The unrotated function corresponding to \((1, 0, 0)\) is \( Y_{1,1}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} = -\sqrt{\frac{3}{8\pi}}(x + iy) \) and the rotated, corresponding to \((0, 0, -1)\) is \(-Y_{1,-1}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi} = -\sqrt{\frac{3}{8\pi}}(x - iy) \).

Rotation around x axis by 180 degrees amounts to \( y \rightarrow -y \) and \( z \rightarrow -z \). Indeed this transforms \( Y_{1,1} \) into \(-Y_{1,-1}\).

2) A hydrogen-like atom with atomic number \( Z \) is in its ground state when, due to nuclear processes (operating at a time scale much shorter than the characteristic time scale of the \( H \) atom), its nucleus is modified to have the atomic number increased by one unit, i.e. to \( Z + 1 \). The electronic state of the atom does not change during this process. What is the probability of finding the atom in the new ground state at a later time? Answer the same question for the new first excited state.

**Ans.:** The hydrogen ground state wave function is

\[
\psi_{1,0,0}(r) = \frac{Z^{3/2}}{\sqrt{\pi a_0^3}} e^{-Zr/a_0}
\]

Once the atomic number is changed, the ground state becomes

\[
\overline{\psi}_{1,0,0}(r) = \frac{(Z + 1)^{3/2}}{\sqrt{\pi a_0^3}} e^{-(Z+1)r/a_0}
\]

and the first excited state becomes

\[
\overline{\psi}_{2,0,0}(r) = \frac{(Z + 1)^{3/2}}{\sqrt{32\pi a_0^3}} (2 - \frac{Z + 1}{a_0}r)e^{-(Z+1)r/(2a_0)}
\]

The probabilities are \( P_1 = (\overline{\psi}_{1,0,0}|\psi_{1,0,0})^2 \) and \( P_2 = (\overline{\psi}_{2,0,0}|\psi_{1,0,0})^2 \).

The evaluation of the radial integrals gives \( P_1 = \frac{(Z(Z+1))^3}{(Z+\frac{1}{2})^6} \) and \( P_2 = \frac{2^{11}}{3^8} (\frac{Z(Z+1)}{Z+\frac{1}{2}})^3 \).
3) Consider the delta-shell potential model, which is a very simple model of the force experienced by a neutron interacting with a nucleus. In this model, the force experienced by neutron has the form

\[ V(r) = -\frac{\hbar^2 g^2}{2\mu} \delta(r - a) \] (8)

Here \( r \) is written in spherical coordinates.

Investigate the existence of bound states in the case of negative energy.

a) Write down the Schroedinger equation for \( u_l(r) \) in spherical coordinates using potential \( V(r) \).

**Ans.:** Schroedinger equation reads

\[ -u'' - g^2 \delta(r - a)u + \frac{l(l + 1)}{r^2} u = -\kappa^2 u \] (9)

where

\[ \kappa = \sqrt{-\frac{2\mu E}{\hbar^2}}. \]

b) What are solutions for free particles (\( V = 0 \))? Which solution can be used for interior part (\( r < a \)) and which for exterior part (\( r > a \))?

**Ans.:** The solution for free particles was given in class, namely spherical bessel and spherical neuman functions. However, these functions are solutions for \( E > 0 \). Here we need bound states, which can be obtained by changing \( kr \rightarrow ikr \) in the argument of the solution.

The solutions are thus

\[ u(r) = A \ r \ j_l(ikr) + B \ r \ n_l(ikr) \] (10)

For small \( r \), only \( j_l(x) \) are well behaved. For large \( r \) we need solution that falls off.

The following large \( x \gg 1 \) expansion of spherical bessel and neuman functions was given in class

\[ j_l(x) \approx \frac{1}{x} \sin(x - l\pi/2) \] (11)

\[ n_l(x) \approx -\frac{1}{x} \cos(x - l\pi/2) \] (12)

For imaginary argument \( ix \), these functions are

\[ j_l(ix) \approx \begin{cases} \frac{\sinh(x)}{x} (-1)^{l/2} & l = 0, 2, 4, \ldots \\ -i\frac{\cosh(x)}{x} (-1)^{(l+1)/2} & l = 1, 3, 5, \ldots \end{cases} \] (13)

\[ n_l(ix) \approx \begin{cases} \frac{i\cosh(x)}{x} (-1)^{l/2} & l = 0, 2, 4, \ldots \\ \frac{\sinh(x)}{x} (-1)^{(l+1)/2} & l = 1, 3, 5, \ldots \end{cases} \] (14)
The following combination of bessel and neuman function falls off in infinity

\[ h_l(ix) = n_l(ix) - ij_l(ix) \propto \frac{e^{-x}}{x} \]  

(15)

This function is also called spherical Henkel function. One can check explicitly

\[ h_l(ix) \approx \begin{cases} 
  i(-1)^{l/2}\frac{e^{-x}}{x} & l = 0, 2, 4, \ldots \\
  (-1)^{(l-1)/2}\frac{e^{-x}}{x} & l = 1, 3, 5, \ldots 
\end{cases} \]  

(16)

Hence, the solution is

\[ u_l(r) = \begin{cases} 
  A r j_l(i\kappa r) & r < a \\
  B r h_l(i\kappa r) & r > a 
\end{cases} \]  

(17)

c) Integrating around the point \( r = a \), determine the discontinuity condition, and hence equation for the eigenstates.

**Ans.:** The integration of the Schroedinger equation gives

\[ u'(a^+) - u'(a^-) = -g^2 u(a) \]  

(18)

We have two boundary conditions: i) continuity at \( r = a \) gives

\[ A a j_l(i\kappa a) = B h_l(i\kappa a) \]  

(19)

and ii) the discontinuity of the Schroedinger equation gives

\[ B a h'_l(i\kappa a) - A a j'_l(i\kappa a) = -g^2 A a j_l(i\kappa a) \]  

(20)

The two equations can be combined together into the following condition

\[ \frac{j'_l(i\kappa a)}{j_l(i\kappa a)} - \frac{h'_l(i\kappa a)}{h_l(i\kappa a)} = \frac{g^2 a}{\kappa a} \]  

(21)

d) Assuming that \( g^2 a = 2 \), solve (possibly numerically) for bound state energy at \( l = 0 \).

**Ans.:** For \( l = 0 \)

\[ j_0(x) = \frac{\sinh(x)}{x} \]  

(22)

\[ h_0(x) = ie^{-x} \]  

(23)

hence the above condition gives

\[ \frac{2}{1 - e^{-2x}} = \frac{g^2 a}{x} \]  

(24)

We are hence looking for the solution of

\[ x = 1 - e^{-2x} \]

for which numerical solution is \( \kappa a = 0.796812 \). The bound state energy hence is

\[ E = -\frac{\hbar^2}{2\mu a^2} (0.796812)^2 \]  

(25)
4) A beam of composite particles is subject to a simultaneous measurement of the spin operators \( S^2 \) and \( S_z \). The measurement gives pairs of values \( s = m_s = 0 \) and \( s = 1, m_s = 1 \) with probabilities \( 3/4 \) and \( 1/4 \) respectively.

(a) Reconstruct the state of the beam immediately before the measurement.

\textbf{Answ.}: Before the measurements, the wave function must have been

\[
|\psi\rangle = \frac{\sqrt{3}}{2} |0,0\rangle + e^{i\alpha} \frac{1}{2} |1, -1\rangle
\]

where \( \alpha \) is any real number.

(b) The particles in the beam with \( s = 1, m_s = 1 \) are separated out and subjected to a measurement of \( S_x \). What are the possible outcomes and their probabilities?

\textbf{Answ.}: Possible outcomes are eigenvalues of \( S_x \) operator for \( s = 1 \) particles. To compute probabilities, we need eigenvectors of operator \( S_x \) (in the \( s = 1 \) sector). The eigenvectors are

\begin{align}
|S_x = +1\rangle &= \frac{1}{2} |1, 1\rangle + \frac{1}{\sqrt{2}} |1, 0\rangle + \frac{1}{2} |1, -1\rangle \\
|S_x = -1\rangle &= \frac{1}{2} |1, 1\rangle - \frac{1}{\sqrt{2}} |1, 0\rangle + \frac{1}{2} |1, -1\rangle \\
|S_x = 0\rangle &= \frac{1}{\sqrt{2}} (|1, 1\rangle - |1, -1\rangle)
\end{align}

The probabilities are then

\begin{align}
P(+1) &= |\langle S_x = +1 | 1, 1 \rangle|^2 = 1/4 \\
P(-1) &= |\langle S_x = -1 | 1, 1 \rangle|^2 = 1/4 \\
P(0) &= |\langle S_x = 0 | 1, 1 \rangle|^2 = 1/2
\end{align}

(c) For the purpose of understanding the symmetry of the wave function, it is convenient to replace spin operators with corresponding orbital angular momentum operators, i.e., \( S_x \rightarrow L_x \) and \( S^2 \rightarrow L^2 \). Write down the spatial wave functions of the states that arise from the second measurement if the operator was orbital angular momentum operator \( L_x \). Give the \( x, y, z \) dependence of such wave functions.

\textbf{Hint}: First figure out the decomposition of the measured states in terms of \( |l, m_l\rangle \) states. Using spherical harmonics, express the resulting wave function in real space.

\textbf{Answ.}: We repeat the decomposition

\begin{align}
|L_x = +1\rangle &= \frac{1}{2} |1, 1\rangle + \frac{1}{\sqrt{2}} |1, 0\rangle + \frac{1}{2} |1, -1\rangle \\
|L_x = -1\rangle &= \frac{1}{2} |1, 1\rangle - \frac{1}{\sqrt{2}} |1, 0\rangle + \frac{1}{2} |1, -1\rangle \\
|L_x = 0\rangle &= \frac{1}{\sqrt{2}} (|1, 1\rangle - |1, -1\rangle)
\end{align}
and use standard expressions for the spherical harmonics, to obtain

\[
\langle r | L_x = \pm 1 \rangle = \sqrt{\frac{3}{8\pi}} \frac{(\pm z - iy/r)}{r} \\
\langle r | L_x = 0 \rangle = -\sqrt{\frac{3}{4\pi}} \frac{x}{r} \tag{35, 36}
\]