Final Exam, Quantum Mechanics 501, Rutgers

December 15, 2015

1. (a) Construct the spin singlet \((S = 0)\) state and the spin triplet \((S = 1)\) states of a two electron system.

**Answ.:** singlet:

\[
|0, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow \downarrow\rangle - |\downarrow \uparrow\rangle)
\]

triplets:

\[
|1, 1\rangle = |\uparrow \uparrow\rangle
\]

\[
|1, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow \downarrow\rangle + |\downarrow \uparrow\rangle)
\]

\[
|1, -1\rangle = |\downarrow \downarrow\rangle
\]

(b) In the experiment we have two electrons, which are in the spin-singlet state. They move in the opposite direction along the \(y\)-axis, and two observers \(A\) and \(B\) measure the spin state of each electron. \(A\) measures the spin component along the \(z\) axis, and \(B\) measures the spin component along an axis making an angle \(\theta\) with the \(z\) axis in the \(xz\)-plane. Suppose that \(A\)’s measurement yields a spin down state and subsequently \(B\) makes a measurement. What is the probability that \(B\)’s measurement yields an up spin (measured along an axis making an angle \(\theta\) with the \(z\)-axis)?

The explicit formula for the representation of the rotation operator \(\text{exp}(-i\mathbf{S} \cdot \hat{n}\theta/\hbar)\) in the spin space is given by the spin 1/2 Wigner matrix

\[
D^{(1/2)}(\hat{n}, \theta) = \begin{pmatrix}
\cos(\theta/2) - in_z \sin(\theta/2) & (-in_x - n_y) \sin(\theta/2) \\
(-in_x + n_y) \sin(\theta/2) & \cos(\theta/2) + in_z \sin(\theta/2)
\end{pmatrix}
\]

and \(\hat{n} = n_x \hat{e}_x + n_y \hat{e}_y + n_z \hat{e}_z\) (\(|\hat{n}| = 1\)) is the axis of rotation.

**Answ.:** Since the state of two electrons is singlet, and we know that the first electron points down, the second has to point up in the same coordinate system. But observer \(B\) is rotated by \(\theta\) around \(y\) axis, hence we need to find how spin-up looks in the rotated coordinate system. We thus apply \(D^{(1/2)}(\hat{e}_y, \theta)\) on \((1, 0)\) to get

\[
|\psi_B\rangle = (\cos(\theta/2), \sin(\theta/2))
\]

The probability for up-spin is thus \(P(|\uparrow\rangle) = \cos^2(\theta/2)\) and for down-spin \(P(|\downarrow\rangle) = \sin^2(\theta/2)\).
2. The Wigner-Eckart theorem is given by

\[ \langle n' j' m' | T^{(l)}_q | n j m \rangle = \langle j' m' | l q, j m \rangle \frac{\langle \langle n' j' | T^{(l)} | n j \rangle \rangle}{\sqrt{2j+1}} \]  

(7)

(a) Explain the meaning of the two terms on the right hand side.

**Answ.**: The first term is the Clebsch-Gordan coefficient, which encodes the geometric properties of the matrix element under rotation. The second is the reduced matrix element, which is a common coefficient for all \( m, m' \) quantum numbers.

(b) The interaction of the electromagnetic field with a charged particle is given by

\[ \Delta H = \frac{e}{2m} A \cdot p \]

If the electromagnetic fields are in the form of a plane wave, then \( A = A_0 \hat{\epsilon} e^{ikr} \), where \( \hat{\epsilon} \) is the polarization of the plane wave. Assuming that the wavelength \( \lambda = 2\pi/k \) is much larger than the atomic size, we may write

\[ A = A_0 \hat{\epsilon}(1 + ik \cdot r + \cdots) \]

such that

\[ \Delta H \approx \frac{e}{2m} A_0 \hat{\epsilon} \cdot p(1 + ik \cdot r) \]

Her we kept both the dipole, and the quadrupole terms.

If the field is polarized along the \( x \)-axis (\( \hat{\epsilon} = \hat{e}_x \)), and the wave propagation is along the \( z \)-axis (\( k = k\hat{e}_z \)) express the Hamiltonian in terms of spherical harmonics. Note that \( p \) is a vector operator, and transforms under rotation as \( r \). For symmetry consideration you may therefore replace \( p \) by \( C r \)

**Answ.**: The Hamiltonian for the above configuration is

\[ \Delta H = \frac{e}{2m} A_0 C(x + ik \cdot xz) \]  

(8)

Using the expressions for \( Y'_{lm} \)s we can get

\[ x = \sqrt{\frac{2\pi}{3}} r(Y_{1,-1} - Y_{1,1}) \]  

(9)

\[ xz = \sqrt{\frac{2\pi}{15}} r^2(Y_{2,-1} - Y_{2,1}) \]  

(10)

hence

\[ \Delta H = \frac{e}{2m} A_0 C \sqrt{\frac{2\pi}{3}} r(Y_{1,-1} - Y_{1,1} + i\frac{kr}{\sqrt{5}}(Y_{2,-1} - Y_{2,1})) \]  

(11)

(c) For the above configuration, derive the selection rules for the dipole and the quadrupole transitions, by considering the transition probability matrix elements \( | \langle \psi_f | \Delta H | \psi_i \rangle |^2 = | \langle l_f m_f | \Delta H | l_i m_i \rangle |^2 \).
The dipole matrix elements are proportional to
\[ \langle l_f m_f | \Delta H_1 | l_i m_i \rangle \propto \langle l_f m_f | Y_{1,-1} - Y_{1,1} | l_i m_i \rangle \]
and
\[ \langle l_f m_f | 1 - 1, l_i m_i \rangle \]
(12)
hence \( |m_f - m_i| = 1 \), and \( |l_f - l_i| \leq 1 \).

The quadrupole matrix elements are
\[ \langle l_f m_f | \Delta H_2 | l_i m_i \rangle \propto \langle l_f m_f | Y_{2,-1} - Y_{2,1} | l_i m_i \rangle \]
and
\[ \langle l_f m_f | 2 - 1, l_i m_i \rangle \]
(13)
hence \( |m_f - m_i| = 1 \), and \( |l_f - l_i| \leq 2 \).

The explicit expressions for the spherical harmonics for \( l = 1, 2 \) are given by
\[
Y_{1,1} = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \frac{x + iy}{r} \\
Y_{1,0} = \frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{z}{r} \\
Y_{2,2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \frac{(x + iy)^2}{r^2} \\
Y_{2,1} = -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \frac{(x + iy)z}{r^2} \\
Y_{2,0} = \frac{1}{4} \sqrt{\frac{5}{\pi}} \frac{2z^2 - x^2 - y^2}{r^2}
\]
and \( Y_{l,-m} = (-1)^m Y_{l,m}^* \).

3. A particle of reduced mass \( \mu = 200 \text{ MeV}/c^2 \) is moving in a spherical potential well of range \( a \) and depth \( V_0 = -150 \text{ MeV} \). \([V(r) = V_0 \text{ for } |r| < a \text{ and } V(r) = 0 \text{ for } |r| > a] \).

The particle is bound in the 1s ground state with binding energy \( E = -5 \text{ MeV} \). (This is supposed to be a very simple model of the deuteron). Note: \( \hbar c = 197.327 \text{ MeV fm} \).

(a) Solve the Schrödinger equation for both \( r < a \) and for \( r > a \).

(b) Using the boundary conditions at \( r = a \), extract the size of the "potential range" \( a \).

(c) Calculate the probability that a measurement of \( r \) will find \( r > a \), i.e. the particle is outside the range of the potential (which is of course forbidden classically).

\textbf{Answ.:} The radial wave function for \( l = 0 \) solution is
\[
\psi(r < a) = A \frac{\sin(kr)}{r} \\
\psi(r > a) = C e^{-\kappa r}
\]
(16)
where
\[ k = \sqrt{\frac{2\mu(E - V_0)}{\hbar^2}} \]
and
\[ \kappa = \sqrt{\frac{2\mu|E|}{\hbar^2}} \]
Given the numbers in the text, we can get
\[ k = 1.22/fm \]
\( \kappa = 0.227/fm \)

The continuity of the wave function and its derivative at \( r = a \) gives the following set of equations

\[
A \sin(ka) = C e^{-\kappa a} \tag{18}
\]
\[
A k \cos(ka) = -C \kappa e^{-\kappa a} \tag{19}
\]

which is satisfied if

\[
\tan(ka) = -\frac{k}{\kappa}. \tag{20}
\]

This equation can be solved for range parameter \( a \), and the first solution (1s) gives:

\[
a = \frac{1}{k} (\pi - \arctan(k/\kappa)) \approx 1.44 \text{ fm}
\]

The probability for the particle to be outside the well is

\[
P(r > a) = \frac{\int_a^\infty |\psi(r)|^2 r^2 dr}{\int_0^\infty |\psi(r)|^2 r^2 dr} \tag{21}
\]

The integration inside the well gives

\[
A^2 \int_0^a \sin^2(kr) dr = A^2 \frac{a}{2} (1 - \frac{\sin(2ka)}{2ka})
\]

and integration outside the box gives

\[
C^2 \int_a^\infty e^{-2\kappa r} dr = C^2 e^{-2\kappa a}. \quad \text{We also have} \quad C/A = e^{\kappa a} \sin(ka)
\]

The ratio that describes the probability \( P(r > a) \) is

\[
\frac{\sin^2(ka)}{\sin^2(ka) + \kappa a (1 - \frac{\sin(2ka)}{2ka})} \approx 0.73 \tag{22}
\]