Ground rules:

- Open book (just one)
- Open notes
- Open homeworks (your own solutions only)
- Write your answer directly on these sheets (continue onto back, if necessary)

There are 5 questions; point assignments are given for each, for a total of 100 points. Do all problems. Pace yourself appropriately. Be sure to check that you have done all parts of all questions. Show your reasoning.

Partial credit will be given. Do as many parts of a problem as possible. If you are stuck on the first part of a problem, try to say something about the later parts if possible.

Feel free to raise your hand to ask a question.

Good luck!!
Problem 1 (20 points)

The four parts of this question are unrelated.

(a) (5 points) Consider the function $f(x) = -\cos(\pi x)$. If $P$ is a projection operator, what kind of operator is $f(P)$? (Give the most specific answer from among: Hermitian, antihermitian, projection, reflection, unitary.) Explain.

(b) (5 points) If $A$ is antihermitian, show that $\sin(A)$ is also antihermitian.

(c) (5 points) Is it true that $U e^A U^\dagger = e^{UAU^\dagger}$ for unitary $U$ and Hermitian $A$? Explain.

(d) (5 points) Let $A$ and $B$ both be reflection operators. Simplify $[A, B]\{A, B\}$ (the product of the commutator and anticommutator) as far as possible.
Problem 2 (20 points)

We shall see later that the operators for angular momentum of a spin-1 particle are

\[ L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

The eigenvectors of \( L_z \) should be pretty obvious. For this problem you also need to know that \( L_x \) has an eigenvalue of +1 with a corresponding eigenvector

\[ |L_x = +1 \rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}. \]

The system starts in the state \( |L_z = 0 \rangle \), then \( L_x \) is measured, and then immediately afterward \( L_z \) is measured. What is the probability that both of the measured values will be +1?
**Problem 3** (20 points)

(a) (7 points) Show that \([P, X^n] = -i\hbar n X^{n-1}\). (Hint: Do by induction.)

(b) (7 points) Show that \([P, f(X)] = -i\hbar f'(X)\) by using (a) and working with the Taylor expansion of \(f(x)\).

(c) (6 points) Show that \([P, f(X)] = -i\hbar f'(X)\) by going to the coordinate-space representation and checking that they give the same result when applied to an arbitrary \(\psi(x)\).
Problem 4 (20 points)

Let \( |\varphi_n\rangle \) be the \( n \)’th eigenstate of a particle of mass \( m \) in an infinite square well extending from \(-a < x < a\), and consider

\[
|\psi\rangle = \sqrt{\frac{2}{3}} |\varphi_1\rangle + \sqrt{\frac{1}{3}} |\varphi_3\rangle
\]

You may assume standard results about the 1D infinite square well problem without rederiving them.

(a) Is \( |\psi\rangle \) a stationary state? Why or why not?

(b) Find the expectation value of the energy in state \( |\psi\rangle \).

(c) If the system is allowed to evolve starting from state \( |\psi\rangle \) at \( t = 0 \), find a later time \( t = T \) at which the system will be in the physical state

\[
\sqrt{\frac{2}{3}} |\varphi_1\rangle - \sqrt{\frac{1}{3}} |\varphi_3\rangle
\]

Hint: To be in the same physical state, the overall phase of the wavefunction does not have to be as given in the equation above.
Problem 5 (20 points)

The three parts of this question all concern the simple harmonic oscillator, but are otherwise unrelated.

(a) (8 points) Compute $\langle 3 | P^3 | 2 \rangle$.

(b) (8 points) Find $\langle 0 | X^n | n \rangle$ for arbitrary $n$, expressing your answer in terms of a factorial.

(c) (4 points) Find $\langle 7 | P^7 | 7 \rangle$. Explain your answer.