Due date: Wednesday, Oct. 2
Reading: Shankar Chs. 2, 3, and 4.1-2

1. [12 points]
   (a) Consider the Hamiltonian $\mathcal{H} = \frac{1}{2m} (p_1^2 + p_2^2) + \frac{a}{2} (x_1 - x_2)^2$ for two particles in 1D, and the four dynamical variables
   
   \begin{align*}
   E_1 &= \frac{1}{2m} p_1^2 + \frac{a}{2} x_1^2 \\
   E_2 &= \frac{1}{2m} p_2^2 + \frac{a}{2} x_2^2 \\
   P &= p_1 + p_2 \\
   W &= x_1 p_2 - x_2 p_1
   \end{align*}

   By computing their Poisson brackets with $\mathcal{H}$, find which of these dynamical variables are constants of the motion.

   (b) Since $P$ is a constant of the motion, we should be able to simplify the problem by writing $\mathcal{H}$ in terms of a set of canonical coordinates and momenta that includes $P$. Show that $x_r = x_2 - x_1$, $X = (x_1 + x_2)/2$, $p_r = (p_2 - p_1)/2$, and $P = p_1 + p_2$ form such a set (by computing their Poisson brackets with each other), and rewrite the Hamiltonian in terms of these center-of-mass and relative variables. Check that $X$ is a cyclic coordinate, consistent with the conservation of $P$.

   (c) Repeat Part (a) for the Hamiltonian $\mathcal{H} = \frac{1}{2m} (p_1^2 + p_2^2) + \frac{a}{2} (x_1^2 + x_2^2)$ and the same four dynamical variables.

   (d) For each constant of the motion found in Part (c), explain physically why it is conserved. When it comes to $W$, show that $W$ is conserved for any Hamiltonian of the form $\mathcal{H} = \frac{1}{2m} (p_1^2 + p_2^2) + f(x_1^2 + x_2^2)$ (i.e., when the potential is only a function of the combination $(x_1^2 + x_2^2)$). Explain this result by noting that the Hamiltonian of a single particle in 2D, $\mathcal{H} = \frac{1}{2m} (p_x^2 + p_y^2) + f(x^2 + y^2)$, takes exactly the same form, and explain the physical meaning of $W$ and why it must be conserved in the latter case.

2. [6 points] The classical Hamiltonian for a particle in an electromagnetic field is given by Shankar Eq. (2.6.2):

   \[ \mathcal{H} = \frac{1}{2m} \left( \mathbf{p} - \frac{q}{c} \mathbf{A}(\mathbf{r},t) \right)^2 + q\phi(\mathbf{r},t) \]

Applying Hamilton’s equations of motion, obtain expressions for $\dot{\mathbf{r}}$ and $\dot{\mathbf{p}}$. Then, taking the time derivative of the first and substituting the second, show that

\[ m\ddot{\mathbf{r}} = q \left( -\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) + \frac{q}{c} \dot{\mathbf{r}} \times (\nabla \times \mathbf{A}) \]

as expected from $\mathbf{F} = q\mathbf{E} + (q/c) \mathbf{v} \times \mathbf{B}$. (Note that it would be incorrect to associate the force with $\dot{\mathbf{p}}$, because $\mathbf{p}$ is not the same as the mechanical momentum $m\mathbf{v}$.)

Problem 3 is on reverse! Turn page over...
3. [12 points] (It is important to understand every aspect of this problem thoroughly.)

Consider two operators whose matrix representations in a 3D vector space are

\[
A = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 2 \\
0 & 2 & 2 \\
\end{pmatrix}
\]

(a) The system is initially in state \(|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle\). If \(A\) were to be measured, what would be the possible allowed value(s) of \(a\) and their probabilities? If \(B\) were to be measured, what would be the possible allowed value(s) of \(b\) and their probabilities?

(b) Compute the expectation value \(\langle \psi | B | \psi \rangle\) for the same state \(|\psi\rangle\) in two ways: (i) from the probabilities computed in part (a); and (ii) directly from the vector-matrix-vector product \(\langle \psi | B | \psi \rangle\).

(c) As a result of the measurement of \(B\) the system is found to have \(b = 4\). What is the new state of the system? In this state, what are the allowed value(s) of \(A\) and their probabilities?

(d) Suppose that at the end of part (c), the measurement of \(A\) yields a value of \(a = -1\). What is the new state of the system?

(e) In general, if \(A\) is measured and found to have value \(a = -1\) when the system is in an unknown initial state, is the final state of the system known after the measurement?

(f) Find a state of the system that is a definite state of both observables \(A\) and \(B\).