1) Consider a system of two non-identical fermions, each with spin $1/2$. One is in a state with $S_{1x} = \frac{\hbar}{2}$, while the other is in a state with $S_{2y} = -\frac{\hbar}{2}$. What is the probability of finding the system in a state with total spin quantum numbers $s = 1$, $m_s = 0$, where $m_s$ refers to the $z$-component of the total spin?

   a) First, find the eigenstate of the operator $S_{1x}$ with the eigenvalue $\frac{\hbar}{2}$. Also find the eigenstate of $S_{2y}$ with the eigenvalue $-\frac{\hbar}{2}$.

   b) Using the rules for summation of angular momenta, find the expression for the state $|s = 1, m_s = 0\rangle$.

   c) Calculate the probability.

2) Consider two spin-1 particles that occupy the state $|s_1 = 1, m_1 = 1; s_2 = 1, m_2 = 0\rangle$.

   What is the probability of finding the system in an eigenstate of the total spin $S^2$ with quantum number $s = 1$? What is the probability for $s = 2$?

3) a) Construct the spin singlet ($S = 0$) state and the spin triplet ($S = 1$) states of a two electron system.

   b) In the experiment we have two electrons, which are in the spin-singlet state. They move in the opposite direction along the $y$-axis, and two observers $A$ and $B$ measure the spin state of each electron. $A$ measures the spin component along the $z$ axis, and $B$ measures the spin component along an axis making an angle $\theta$ with the $z$ axis in the $xz$-plane. Suppose that $A$’s measurement yields a spin down state and subsequently $B$ makes a measurement. What is the probability that $B$’s measurement yields an up spin (measured along an axis making an angle $\theta$ with the $z$-axis)?

The explicit formula for the representation of the rotation operator $\exp(-iS \cdot \hat{n}\theta/\hbar)$ in the spin space is given by the spin 1/2 Wigner matrix

\[
D^{(1/2)}(\hat{n}, \theta) = \begin{pmatrix}
\cos(\theta/2) - in_z \sin(\theta/2) & (-in_x - n_y) \sin(\theta/2) \\
(-in_x + n_y) \sin(\theta/2) & \cos(\theta/2) + in_z \sin(\theta/2)
\end{pmatrix}
\]

(1)

and $\hat{n} = n_x \hat{e}_x + n_y \hat{e}_y + n_z \hat{e}_z$ ($|\hat{n}| = 1$) is the axis of rotation.
4) The Wigner-Eckart theorem is given by

\[ \langle n'j'm'|T_l^{(l)}|njm \rangle = \langle j'm'|T_l|njm \rangle \frac{\langle \langle n'j'|T_l|n \rangle \rangle}{\sqrt{2j+1}} \]  

(a) Explain the meaning of the two terms on the right hand side.

(b) The interaction of the electromagnetic field with a charged particle is given by

\[ \Delta H = \frac{e}{2m} \mathbf{A} \cdot \mathbf{p} \]

If the electromagnetic fields are in the form of a plane wave, then \( \mathbf{A} = A_0 \hat{\epsilon} e^{i\mathbf{k}\cdot\mathbf{r}} \), where \( \hat{\epsilon} \) is the polarization of the plane wave. Assuming that the wavelength \( \lambda = 2\pi/k \) is much larger than the atomic size, we may write

\[ \mathbf{A} = A_0 \hat{\epsilon}(1 + i\mathbf{k} \cdot \mathbf{r} + \cdots) \]

such that

\[ \Delta H \approx \frac{e}{2m} A_0 \hat{\epsilon} \cdot \mathbf{p}(1 + i\mathbf{k} \cdot \mathbf{r}) \]

Here we kept both the dipole (the first term), and the quadrupole terms (the second term).

If the field is polarized along the \( x \)-axis (\( \hat{\epsilon} = \hat{e}_x \)), and the wave propagation is along the \( z \)-axis (\( \mathbf{k} = k\hat{e}_z \)) express the Hamiltonian in terms of spherical harmonics. Note that \( \mathbf{p} \) is a vector operator, and transforms under rotation as \( \mathbf{r} \). For symmetry consideration you may therefore replace \( \mathbf{p} \) by \( C\mathbf{r} \).

(c) For the above configuration, derive the selection rules for the dipole and the quadrupole transitions, by considering the transition probability matrix elements

\[ |\langle \psi_f|\Delta H|\psi_i \rangle|^2 = |\langle l_f m_f |\Delta H | l_i m_i \rangle|^2 \]. Note: selection rules state under which conditions is a transition possible.

The explicit expressions for the spherical harmonics for \( l = 1, 2 \) are given by

\[ Y_{1,1} = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \frac{x + iy}{r} \quad Y_{1,0} = \frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{z}{r} \]

(3)

\[ Y_{2,2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \frac{(x + iy)^2}{r^2} \quad Y_{2,1} = -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \frac{(x + iy)z}{r^2} \quad Y_{2,0} = \frac{1}{4} \sqrt{\frac{5}{\pi}} \frac{2z^2 - x^2 - y^2}{r^2} \]

(4)

and \( Y_{l,-m} = (-1)^m Y_{l,m}^* \).