1. (a) Construct the spin singlet \((S = 0)\) state and the spin triplet \((S = 1)\) states of a two electron system.

(b) In the experiment we have two electrons, which are in the spin-singlet state. They move in the opposite direction along the \(y\)-axis, and two observers \(A\) and \(B\) measure the spin state of each electron. \(A\) measures the spin component along the \(z\) axis, and \(B\) measures the spin component along an axis making an angle \(\theta\) with the \(z\) axis in the \(xz\)-plane. Suppose that \(A\)'s measurement yields a spin down state and subsequently \(B\) makes a measurement. What is the probability that \(B\)'s measurement yields an up spin (measured along an axis making an angle \(\theta\) with the \(z\)-axis)?

The explicit formula for the representation of the rotation operator \(\exp(-iS \cdot \hat{n} \theta/\hbar)\) in the spin space is given by the spin 1/2 Wigner matrix

\[
D^{(1/2)}(\hat{n}, \theta) = \begin{pmatrix}
\cos(\theta/2) - in_z \sin(\theta/2) \\
(-in_x - n_y) \sin(\theta/2) \\
(-in_x + n_y) \sin(\theta/2) \\
\cos(\theta/2) + in_z \sin(\theta/2)
\end{pmatrix}
\]

\[\text{(1)}\]

and \(\hat{n} = n_x \hat{e}_x + n_y \hat{e}_y + n_z \hat{e}_z (|\hat{n}| = 1)\) is the axis of rotation.

2. The Wigner-Eckart theorem is given by

\[
\langle n' j' m'|T_q^{(l)}|njm \rangle = \langle j' m'|lq, jm \rangle \frac{\langle n' j'|T_q^{(l)}|nj \rangle}{\sqrt{2j + 1}}
\]

\[\text{(2)}\]

(a) Explain the meaning of the two terms on the right hand side.

(b) The interaction of the electromagnetic field with a charged particle is given by

\[
\Delta H = \frac{e}{2m} \mathbf{A} \cdot \mathbf{p}
\]

If the electromagnetic fields are in the form of a plane wave, then \(\mathbf{A} = A_0 \hat{\varepsilon} e^{i k r}\), where \(\hat{\varepsilon}\) is the polarization of the plane wave. Assuming that the wavelength \(\lambda = 2\pi/k\) is much larger than the atomic size, we may write

\[
\mathbf{A} = A_0 \hat{\varepsilon}(1 + ik \cdot r + \cdots)
\]

such that

\[
\Delta H \approx \frac{e}{2m} A_0 \hat{\varepsilon} \cdot \mathbf{p}(1 + ik \cdot r)
\]
Her we kept both the dipole (the first term), and the quadrupole terms (the second term).

If the field is polarized along the $x$-axis ($\hat{\varepsilon} = \hat{e}_x$), and the wave propagation is along the $z$-axis ($\mathbf{k} = k\hat{e}_z$) express the Hamiltonian in terms of spherical harmonics. Note that $\mathbf{p}$ is a vector operator, and transforms under rotation as $\mathbf{r}$. For symmetry consideration you may therefore replace $\mathbf{p}$ by $C\mathbf{r}$

(c) For the above configuration, derive the selection rules for the dipole and the quadrupole transitions, by considering the transition probability matrix elements

$$|\langle \psi_f | \Delta H | \psi_i \rangle|^2 = |\langle l_f m_f | \Delta H | l_i m_i \rangle|^2.$$ Note: selection rules state under which conditions is a transition possible.

The explicit expressions for the spherical harmonics for $l=1,2$ are given by

$$Y_{1,1} = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \frac{x+iy}{r} \quad Y_{1,0} = \frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{z}{r} \quad (3)$$

$$Y_{2,2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \frac{(x+iy)^2}{r^2} \quad Y_{2,1} = -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \frac{(x+iy)z}{r^2} \quad Y_{2,0} = \frac{1}{4} \sqrt{\frac{5}{\pi}} \frac{2z^2 - x^2 - y^2}{r^2} \quad (4)$$

and $Y_{l,-m} = (-1)^m Y_{l,m}^*.$

3. A particle of reduced mass $\mu = 200 \text{ MeV}/c^2$ is moving in a spherical potential well of range $a$ and depth $V_0 = -150 \text{ MeV}$. [$V(\mathbf{r}) = V_0$ for $|\mathbf{r}| < a$ and $V(\mathbf{r}) = 0$ for $|\mathbf{r}| > a$]. The particle is bound in the $1s$ ground state with binding energy $E = -5 \text{ MeV}$. (This is supposed to be a very simple model of the deuteron). Note: $\hbar c = 197.327 \text{ MeV fm}.$

(a) Solve the Schroedinger equation for both $r < a$ and for $r > a$.

(b) Using the boundary conditions at $r = a$, extract the size of the ’potential range’ $a$.

(c) Calculate the probability that a measurement of $r$ will find $r > a$, i.e. the particle is outside the range of the potential (which is of course forbidden classically).