FOUR-DIMENSIONAL WALL-CROSSING FROM THREE-DIMENSIONAL FIELD THEORY

WORK DONE WITH

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BASED ON

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OUTLINE

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1. INTRODUCTION

THIS TALK IS ABOUT THE BPS SPECTRUM OF $\mathcal{N}=2$, $D=4$ FIELD THEORIES.

THE BPS SPECTRUM OF THE THEORY ON $\mathbb{R}^4$ IS A "PIECEWISE CONSTANT" FUNCTION OF THE BOUNDARY CONDITIONS AT ∞.

RECENTLY THERE HAS BEEN SOME PROGRESS IN UNDERSTANDING PRECISELY <u>HOW</u> THE SPECTRUM DEPENDS ON BOUNDARY CONDITIONS.

THESE ARE CALLED WALL-CROSSING FORMULAE (WCF). THIS TALK WILL GIVE A PHYSICAL INTERPRETATION AND PROOF OF A FAMOUS WCF OF KONTSEVICH + SOIBELMAN.

2. REVIEW $\mathcal{N}=2, D=4$ WALL CROSSING

CONSIDER A THEORY ON $\mathbb{R}^4$
WITH $\mathcal{N}=2$ SUPERPOINCARÉ SYMMETRY $\mathcal{S}$

LET $\mathcal{H}$ BE THE ONE-PARTICLE HILBERT SPACE.

AS A REPRESENTATION OF $\mathcal{S}$, $\mathcal{H}$ DEPENDS ON THE BOUNDARY CONDITIONS OF FIELDS AT ∞.

THESE BOUNDARY CONDITIONS ARE VALUED IN THE MODULI SPACE OF VACUA: $\mathcal{M}_V$.

FOR $u \in \mathcal{M}_V$, WRITE $\mathcal{H}_u$.

FOR ALL $u \in \mathcal{M}_V$ THERE IS AN UNBROKEN ABELIAN GAUGE SYMMETRY OF RANK r, SO $\mathcal{H}$ IS GRADED BY THE SYMPLECTIC LATTICE Γ OF ELEC. + MAG. CHARGES. (OF RANK $2r$).

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma,u}$$

ON EACH SUBSPACE $\mathcal{H}_{\gamma,u}$ THE CENTRAL CHARGE OPERATOR $Z \in \Delta$ IS A SCALAR.

DENOTE THE VALUE $Z_\gamma(u)$

DEF. $\mathcal{H}^{BPS}_{\gamma,u}$ = SUBSPACE SATURATING THE BPS BOUND.

ON THIS SUBSPACE $E = |Z_\gamma(u)|$

SOME BPS PARTICLES CAN BE VIEWED AS BOUNDSTATES OF OTHERS: DECAY WHEN BOUNDSTATE ENERGY $E(u) \longrightarrow 0$.

[Cecotti et.al. ; Seiberg & Witten]

$Z_\gamma(u)$ IS LINEAR IN $\gamma = \gamma_1 + \gamma_2$ SO

$$E(u) = |Z_\gamma(u)| - \left(|Z_{\gamma_1}(u)| + |Z_{\gamma_2}(u)|\right) \leq 0$$

$\Longrightarrow$ DECAY ONLY HAPPENS ALONG WALLS OF MARGINAL STABILITY:

$$MS(\gamma_1, \gamma_2) := \{ u \mid Z_{\gamma_1}(u)/Z_{\gamma_2}(u) \in \mathbb{R}_+ \}$$

FOR DECAYS OF THE FORM

$$\gamma = N_1 \gamma_1 + N_2 \gamma_2 \qquad N_1, N_2 > 1$$

THE METHODS OF DENEF-MOORE (LECTURE I) ARE DIFFICULT TO USE.

KONTSEVICH & SOIBELMAN PROPOSED A REMARKABLE FORMULA FOR THE CHANGE OF THE INDEX:

$$\Omega(\gamma; u) := -\frac{1}{2} \mathrm{Tr}_{\mathcal{H}^{BPS}_{\gamma,u}} (2J_3)^2 (-1)^{2J_3}$$

WHICH INCLUDES ALL CASES.

[IN SUPERGRAVITIES ARISING FROM CY COMPACTIFICATION $\Omega(\gamma; u) =$ "GENERALIZED DONALDSON-THOMAS INVT."]

3. THE KONTSEVICH–SOIBELMAN FORMULA

FOR THE LATTICE Γ OF CHARGES INTRODUCE A LIE ALGEBRA WITH ONE GENERATOR e_γ FOR EACH $\gamma \in \Gamma$:

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1,\gamma_2\rangle\, e_{\gamma_1+\gamma_2}$$

EXPONENTIATE TO FORM A GROUP.

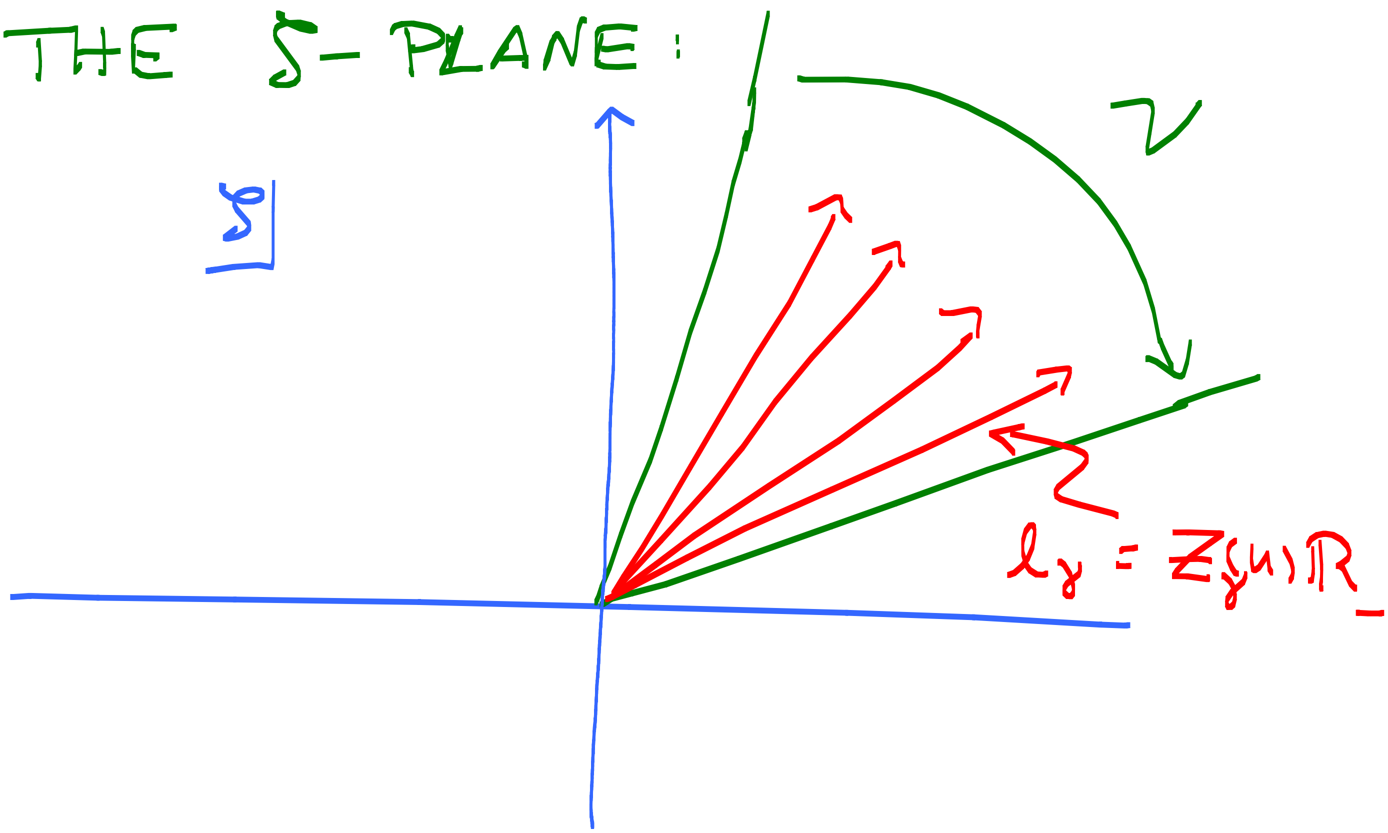
DEFINE A GROUP ELEMENT:

$$U_\gamma := \exp\left(\sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right)$$

TO EACH $\gamma \in \Gamma$ ASSOCIATE THE "BPS RAY" IN THE COMPLEX PLANE

$$\ell_\gamma := \{ \zeta \mid \zeta \in Z_\gamma(u) \cdot \mathbb{R}_- \}$$

CHOOSE A CONVEX CONE $\mathcal{V}$ IN THE ζ-PLANE:



AND CONSIDER THE BPS RAYS WITHIN THE CONE $\mathcal{V}$

ASSOCIATE THE GROUP ELEMENT

$$\mathcal{A}_V = \prod^{\rightarrow}_{\ell_\gamma \subset V} U_\gamma^{\Omega(\gamma;u)}$$

WITH THE PRODUCT TAKEN OVER THE RAYS IN THE <u>CLOCKWISE ORDER</u> (DECREASING SLOPE)

FOR LATER CONVENIENCE

NOTE THAT IF $\gamma_1 = N\gamma_2$, $N>0$

THEN $\ell_{\gamma_1} = \ell_{\gamma_2}$, SO DEFINE

$$S_\gamma := \prod_{\ell_{\gamma'} = \ell_\gamma} U_{\gamma'}^{\Omega(\gamma';u)}$$

SO:

$$\mathcal{A}_V = \prod^{\rightarrow}_{\substack{\gamma \text{ PRIM} \\ \ell_\gamma \subset V}} S_\gamma$$

NOTE THAT

$$A_\nu = \overrightarrow{\prod_{\ell_\gamma \subset \nu}} U_\gamma^{\Omega(\gamma; u)}$$

DEPENDS ON u IN TWO WAYS

1. THE ORDERING OF FACTORS DEPENDS ON u

2. THE $\Omega(\gamma; u)$ DEPEND ON u ...

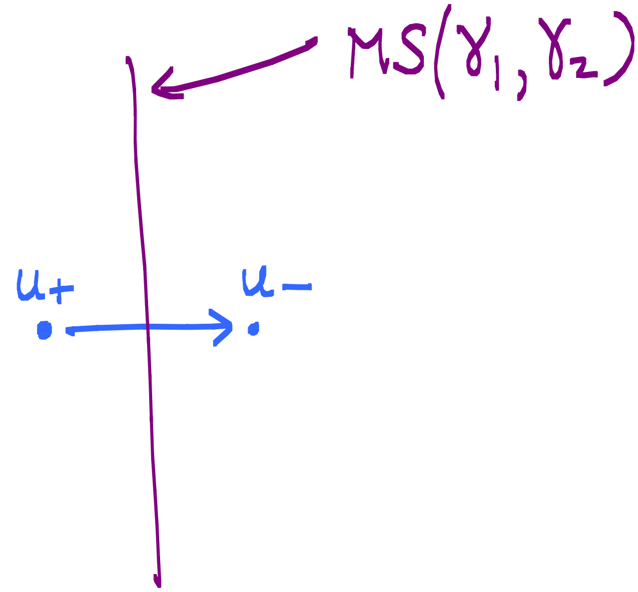
THE KS FORMULA STATES THAT

$$A_{\nu} = \overrightarrow{\prod_{\ell_\gamma \subset \nu}} U_\gamma^{\Omega(\gamma;u)}$$

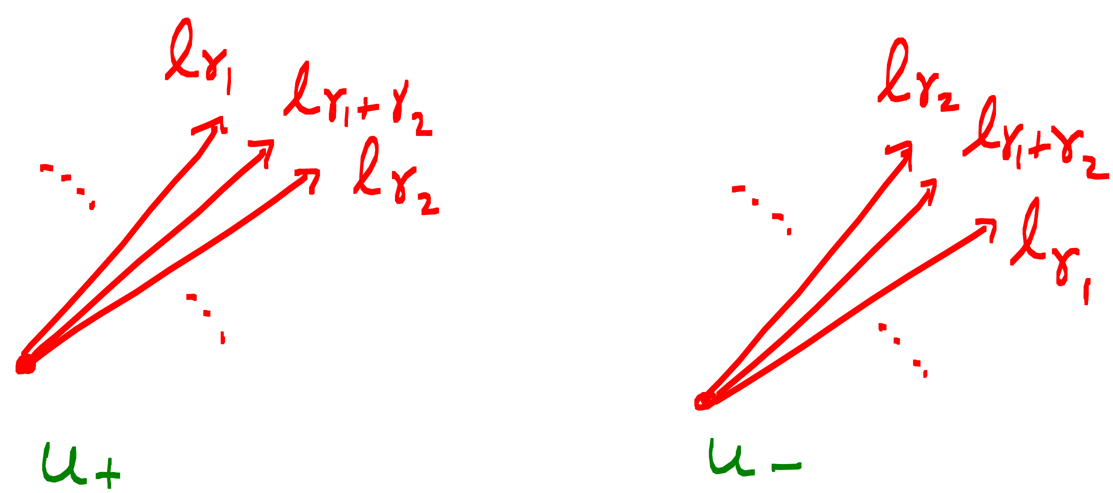
IS <u>CONSTANT IN u</u> AS LONG AS NO BPS RAY ENTERS OR LEAVES THE SECTOR ν.

THIS IS A WALL-CROSSING FORMULA ...

AS u CROSSES A WALL



$$\ell_\gamma = Z_\gamma(u) \cdot \mathbb{R}_- \quad \text{ROTATES}$$



$\Longrightarrow$ EXCHANGE ORDER IN

$$A_\nu = \prod_{\ell_\gamma \subset \nu}^{\rightarrow} U_\gamma^{\Omega(\gamma; u)}$$

AT A GENERIC POINT $t \in MS(\gamma_1, \gamma_2)$

$$Z(\gamma;t) \parallel Z_1, Z_2 \implies$$

$$\gamma = \gamma_{a,b} = a\gamma_1 + b\gamma_2$$

$(\gamma_1, \gamma_2$ primitive$)$

FOR SMALL CONE ANGLE ONLY THE LIE SUBALGEBRA $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ CONTRIBUTES:

$$[e_{a,b}, e_{c,d}] = (-1)^{(ad-bc)I_{12}} (ad-bc) I_{12}\, e_{a+c,b+d}$$

$$U_{a,b} := \exp\left(\sum_{m=1}^{\infty} \frac{e_{ma,mb}}{m^2}\right)$$

$$\prod_{\substack{a/b \uparrow \\ a \geq 0}} U_{a,b}^{\Omega^-(\gamma_{a,b})} = \prod_{\substack{a/b \downarrow \\ a \geq 0}} U_{a,b}^{\Omega^+(\gamma_{a,b})}$$

ONE CAN RECOVER THE PRIMITIVE & SEMI-PRIMITIVE W.C.F. FROM THIS FORMULATION...

[with Wu-yen Chuang]

LIE ALGEBRA IS FILTERED $\Longrightarrow$

CAN RESTRICT TO

Heisenberg Algebra
$$\left\{\begin{array}{l}[e_{0,1}, e_{1,0}] = (-1)^{I_{12}-1} I_{12}\, e_{1,1} \\ e_{1,1} \ \underline{\text{CENTRAL}}\end{array}\right.$$

$$U_{0,1}^{\Omega^-(\gamma_1)}\, U_{1,1}^{\Omega^-(\gamma_1+\gamma_2)}\, U_{1,0}^{\Omega^-(\gamma_2)}$$

$$= U_{1,0}^{\Omega^+(\gamma_2)}\, U_{1,1}^{\Omega^+(\gamma_1+\gamma_2)}\, U_{0,1}^{\Omega^+(\gamma_1)}$$

$$\boxed{U_{01} U_{10} = U_{1,1}^{\pm I_{12}} \cdot U_{10} U_{01}} \Longrightarrow$$

$$U_{1,1}^{\Omega^+(\gamma_1+\gamma_2)-\Omega^-(\gamma_1+\gamma_2)} = U_{0,1}^{\Omega(\gamma_1)} U_{1,0}^{\Omega(\gamma_2)} U_{0,1}^{-\Omega(\gamma_1)} U_{1,0}^{-\Omega(\gamma_2)}$$

$$= U_{1,1}^{I_{12}\,\Omega(\gamma_1)\,\Omega(\gamma_2)}$$

$\Longrightarrow$ PRIMITIVE W.C. FORMULA!

RELATION TO SYMPLECTIC TRANSFORMATIONS

THE KS LIE ALGEBRA IS ISOMORPHIC TO A LIE ALGEBRA OF SYMPLECTIC TRANSFORMATIONS OF A TORUS, ALBEIT <u>NOT</u> CANONICALLY:

- INTRODUCE THE SYMPLECTIC TORUS

$$T = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^*$$

$$\gamma \in \Gamma \Rightarrow \text{FUNCTION } X_\gamma : T \to \mathbb{C}^*$$

- CHOOSING A BASIS γ_i, DEFINE:

$$\varpi = \frac{1}{2} \epsilon^{ij} \frac{dX_i}{X_i} \wedge \frac{dX_j}{X_j} \qquad , \qquad \epsilon_{ij} = \langle \gamma_i, \gamma_j \rangle$$

- CHOOSE A QUADRATIC REFINEMENT

$$\frac{\sigma(\gamma_1+\gamma_2)}{\sigma(\gamma_1)\sigma(\gamma_2)} = (-1)^{\langle\gamma_1,\gamma_2\rangle}, \qquad \sigma = \pm 1$$

- $\sigma(\gamma)e_\gamma$ GENERATE THE LIE ALGEBRA OF SYMPLECTIC VECTOR FIELDS

- THE GROUP ELEMENT U_γ IS THE SYMPLECTOMORPHISM

$$\boxed{U_\gamma : X_{\gamma'} \longrightarrow X_{\gamma'}\left(1-\sigma(\gamma)X_\gamma\right)^{\langle\gamma',\gamma\rangle}}$$

EXAMPLE

$$r=1 \implies \Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$\langle (a,b),(a',b') \rangle = ab' - a'b$$

$$T \cong \mathbb{C}^* \times \mathbb{C}^*$$

$$x = X_{1,0} \qquad y = X_{0,1}$$

$$\overline{\omega} = \frac{dx}{x} \wedge \frac{dy}{y}$$

$$U_{a,b} : \begin{cases} x \longrightarrow x\left(1-(-1)^{ab}x^a y^b\right)^b \\ y \longrightarrow y\left(1-(-1)^{ab}x^a y^b\right)^{-a} \end{cases}$$

4. COMPACTIFICATION OF $N=2, D=4$ FIELD THEORIES

A. SEIBERG-WITTEN SOLUTION

G - COMPACT S.S. GAUGE GROUP, RANK $= r$

$\Longrightarrow$ $D=4, N=2$ FIELD THEORY

(CAN ALSO INCLUDE HM's)

$$\mathcal{M}_V = (\mathfrak{g}_{\mathbb{C}})^G \cong \mathbb{C}^r \quad \left(u_2 = \mathrm{Tr}\,\Phi^2,\ u_3 = \mathrm{Tr}\,\Phi^3, \dots \right)$$

S & W GAVE FORMULAE FOR

- $Z_\gamma(u)$
- LOW ENERGY ABELIAN GAUGE THRY.

IN TERMS OF

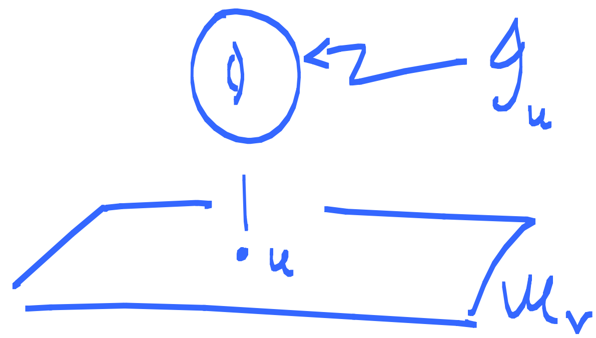
SPECIAL KÄHLER GEOMETRY

REVIEW SPECIAL KÄHLER GEOM:

c.f. D. FREED, hep-th/9712042

VIEW Γ AS A LOCAL SYSTEM OVER $\mathcal{M}_v$

$$\begin{tikzcd} \mathcal{J}_u \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ u \arrow[r, hook] & \mathcal{M}_v \end{tikzcd}$$



$$\mathcal{J} = \Gamma^* \underset{\mathbb{Z}}{\otimes} \left(\mathbb{R}/2\pi\mathbb{Z}\right) \cong U(1)^{2r}$$

FIBERS = ABELIAN VARIETIES

IN REGIONS OF $\mathcal{M}_v$ CHOOSE A DUALITY FRAME:

$$\Gamma = \Gamma_{el} \oplus \Gamma_{mag} \,, \quad \Gamma_{mag} = \Gamma_{el}^*$$

$$= \mathrm{Span}\{\alpha_I\} \oplus \mathrm{Span}\{\beta^I\}$$

$$\langle \alpha_I, \alpha_J \rangle = \langle \beta^I, \beta^J \rangle = 0 \qquad \langle \alpha^I, \beta^J \rangle = \delta_I{}^J$$

CHOOSING A DUALITY FRAME,

Σ_u HAS PERIOD MATRIX τ_{IJ}

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1. LOW ENERGY LAGRANGIAN:

$$\mathcal{L} = \frac{-1}{4\pi} \operatorname{Im}\tau_{IJ} \left(da^I * d\bar{a}^J + F^I * F^J \right)$$

$$+ \frac{1}{4\pi} \operatorname{Re} \tau_{IJ} \; F^I \wedge F^J$$

$$a^I = Z_{\alpha_I}(u) \qquad I = 1, \dots, r$$

LOCAL COORD'S ON $\mathcal{M}_V$

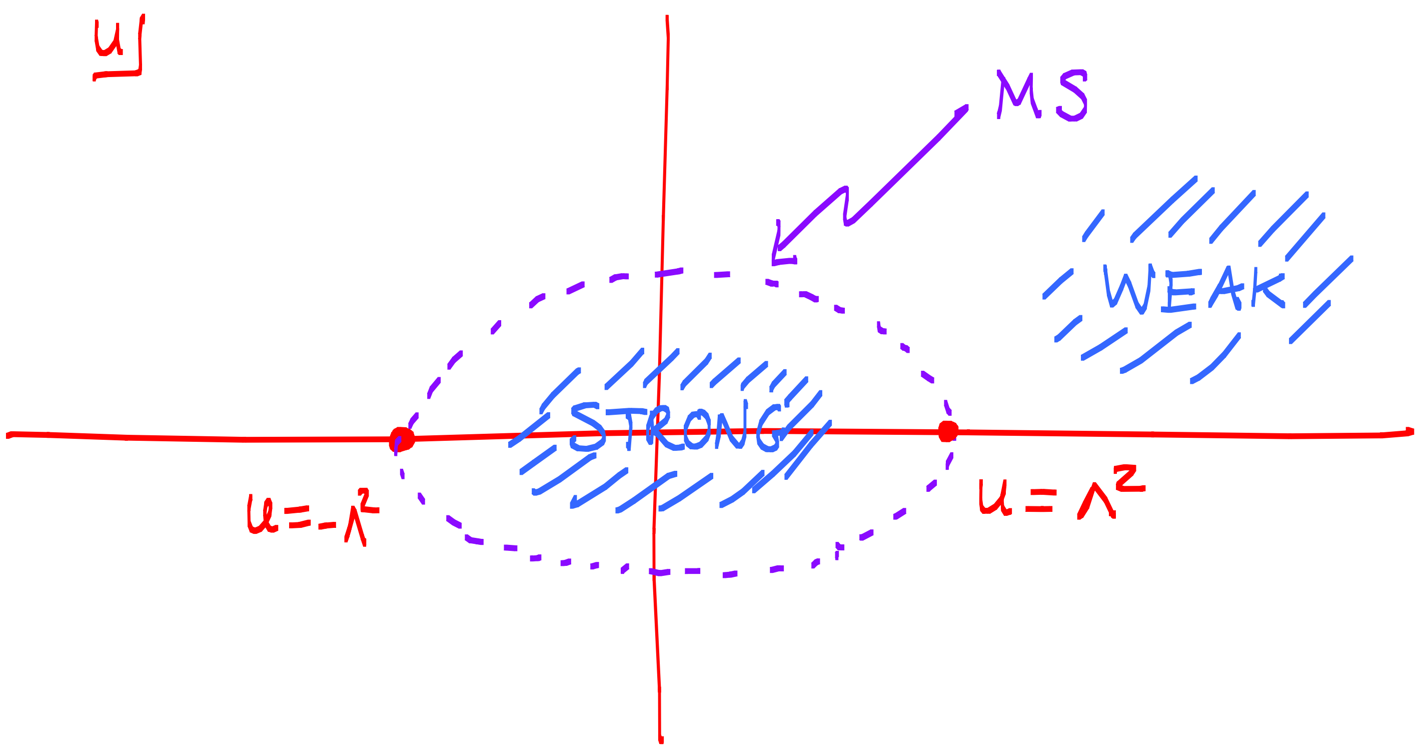
2. CENTRAL CHARGE FUNCTION

$$\boxed{\begin{aligned} Z_\gamma(u) &= a \cdot \gamma_{el} + a_D \cdot \gamma_{mg} \\ \tau_{IJ} &= \frac{\partial a_{DI}}{\partial a^J} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J} \end{aligned}}$$

S & W IDENTIFY $\mathcal{J}_u$ AS JACOBIANS OF AN EXPLICIT FAMILY OF RIEMANN SURFACES

BASIC EXAMPLE: $G = SU(2)$

$$\Sigma_u: \quad y + \frac{\Lambda^4}{y} = x^2 - 2u$$



$$a = \oint_\alpha x \frac{dy}{y} \qquad a_D = \oint_\beta x \frac{dy}{y}$$

SPECTRUM:

$$\mathcal{H}^{BPS}_{WEAK} = \bigoplus_{n\in\mathbb{Z}} HM(2n,1) \oplus VM(2,0)$$

$\oplus$ CONJUGATE

$$\mathcal{H}^{BPS}_{STRONG} = HM(2,-1) \oplus HM(0,1)$$

$\oplus$ CONJUGATE

[Bilal & Ferrari]

KS IDENTITY:

$$U_{2,-1}U_{0,1} = U_{0,1}U_{2,1}U_{4,1}\cdots U_{2,0}^{-2}\cdots U_{6,-1}U_{4,-1}U_{2,-1}$$

IT IS TRUE !!!

B. COMPACTIFY ON A CIRCLE.

- NOW CONSIDER THE THEORY ON $\mathbb{R}^3 \times S^1_R$.

- LOW ENERGY THEORY IS A 3D σ-MODEL: $\mathbb{R}^3 \longrightarrow \mathcal{M}$

$$a^I(\vec{x}, x^4) \longrightarrow a^I(\vec{x})$$

$$\left.\begin{aligned} \varphi_e^I &= \int_{S^1} A_4^I \, dx^4 \\ \varphi_{m,I} &= \int_{S^1} (A_{D,4})_I \, dx^4 \end{aligned}\right\} \text{PERIODIC!}$$

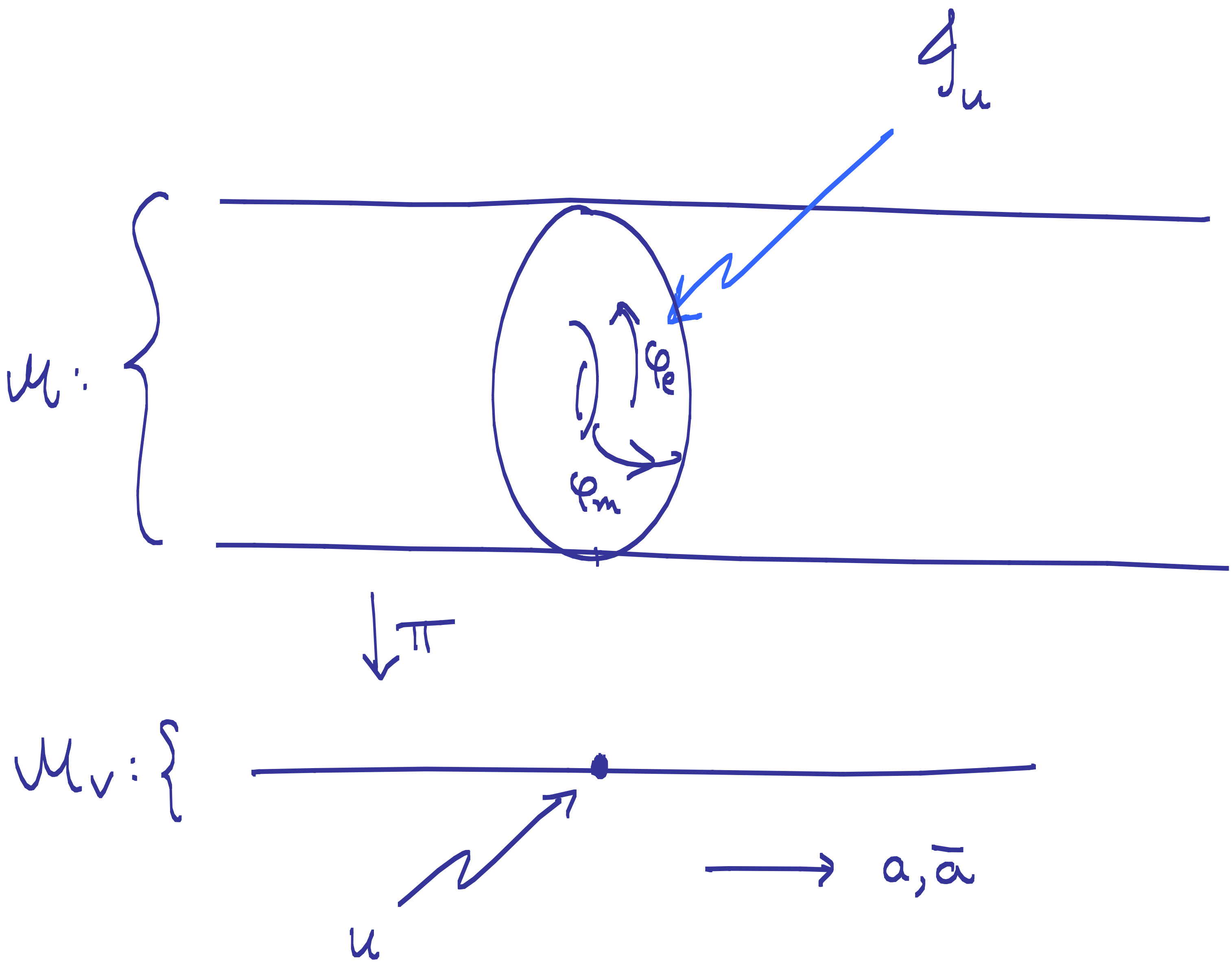
- SUPERSYMMETRY $\Longrightarrow$

$\mathcal{M}$ MUST CARRY A HYPERKÄHLER METRIC

LET US TRY TO DESCRIBE IT

<u>TOPOLOGICALLY</u> $\mathcal{M}$ IS A TORUS FIBRATION OVER $\mathcal{M}_V$:

IT IS EXACTLY $\mathcal{M} = \mathcal{J} = \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z}$ THAT APPEARED ABOVE:



THE SEMI-FLAT METRIC

LEADING $R \to \infty$ APROXIMATION:

USE DIMENSIONAL REDUCTION

+ DUALIZATION OF 3D GAUGE FIELD:

$$\mathcal{L}^{(3)} = -\frac{R}{2} \operatorname{Im}\tau_{IJ}\, da^I * d\bar{a}^J$$

$$-\frac{1}{8\pi^2 R} (\operatorname{Im}\tau)^{-1,IJ}\, dz_I * d\bar{z}_J$$

$$dz_I = d\varphi_{m,I} - \tau_{IJ}\, d\varphi_e^J$$

THIS DEFINES THE SEMIFLAT METRIC

$$g^{sf} = R(\operatorname{Im}\tau)|da|^2 + \frac{1}{4\pi^2 R}(\operatorname{Im}\tau)^{-1}|dz|^2$$

C. THE KEY IDEA

- THE METRIC g^{sf} RECEIVES QUANTUM CORRECTIONS FROM BPS PARTICLE WORLD-LINES WRAPPING S^1.

- THEREFORE THE QUANTUM CORRECTIONS DEPEND ON THE BPS SPECTRUM.

- THE TRUE METRIC g SHOULD BE A SMOOTH METRIC ON $\mathcal{M}$ AWAY FROM THE LOCUS IN $\mathcal{M}_V$ WHERE BPS PARTICLES BECOME $M=0$.

- SMOOTHNESS OF g ACROSS WALLS OF M.S. IMPLIES A WCF.

CLAIM: IT IS THE KS WCF.

5. TWISTOR SPACE APPROACH

WE WILL USE HITCHIN'S THEOREM: KNOWING $(\mathcal{M}, g)$ IS EQUIVALENT TO KNOWING TWISTOR SPACE $Z := \mathcal{M} \times \mathbb{CP}^1$ AS A HOLOMORPHIC MANIFOLD.

<u>THEOREM</u>: IF $\mathcal{M}$ IS HK OF DIMENSION $4r$ THEN:

1. $\exists$ HOLO. FIBRATION

$$p : Z \longrightarrow \mathbb{CP}^1$$

$M_\zeta = p^{-1}(\zeta) =$ M IN COMPLEX STRUCTURE ζ

2. $\exists$ HOLOMORPHIC SECTION

ϖ OF $\Omega^2_{Z/\mathbb{CP}^1} \otimes \mathcal{O}(2)$

$\varpi|_{M_\zeta} =$ HOLOMORPHIC SYMPLECTIC FORM ON M_ζ

3. $\forall x \in M$, $\exists$ HOLOMORPHIC SECTION $s_x : \mathbb{CP}^1 \longrightarrow Z$ WITH NORMAL BUNDLE $\mathcal{O}(1)^{\oplus 2r}$

4. $\exists$ ANTI-HOLOMORPHIC $\sigma : Z \to Z$ COVERING $\zeta \to -1/\bar{\zeta}$

GIVEN 1, 2, 3, 4 ONE CAN RECONSTRUCT THE METRIC:

FOR $\zeta \in \mathbb{C}^*$:

$$\varpi = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\,\omega_-$$

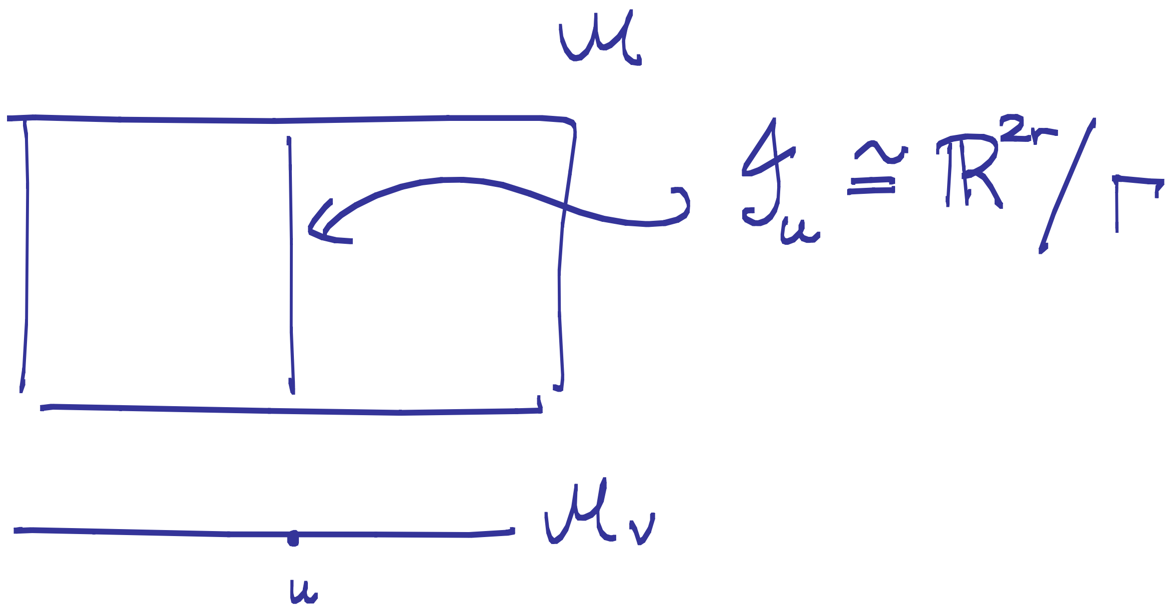
KÄHLER FORM

$$\omega_+ = \omega_1 + i\omega_2$$

OUR STRATEGY IS TO CONSTRUCT THE HOLOMORPHIC SECTIONS S_x AND ϖ

EXPLICITLY FOR THE 3D σ-MODEL TARGET.

USE THE TORUS FIBRATION OF $\mathcal{M}$:



- FOR $\zeta = 0$, $\mathcal{J}_u$ IS HOLOMORPHIC
- FOR $\zeta \neq 0, \infty$ $\mathcal{J}_u$ IS NEITHER HOLO. NOR ANTI-HOLO.

- HOLOMORPHIC FUNCTION ON $\mathcal{M}_\zeta$ IS DETERMINED BY RESTRICTION TO SOME (i.e. ANY) FIBER $\mathcal{J}_{u_0}$ AS A C^∞ FUNCTION.

• A BASIS OF C^∞ FUNCTIONS ON THE TORUS $\mathcal{J}_{u_0}$ IS LABELLED BY $\gamma \in \Gamma_{u_0}$

$$\theta_\gamma : \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$$

• CORRESPONDING <u>HOLOMORPHIC</u> FUNCTION ON $\mathcal{M}_\zeta$

$$\mathcal{X}_\gamma = \exp\left[i\theta_\gamma + \cdots \right]$$

THE COLLECTION OF FUNCTIONS $\mathcal{X}_\gamma$ DEFINES A SINGLE MAP

$$\mathcal{X} : \mathcal{J} \longrightarrow T = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^*$$

NOW <u>DEFINE</u> $\quad \tilde{\varpi} := \mathcal{X}^*(\varpi^T)$

$$\boxed{\text{CONSTRUCTING HK METRIC} \Longleftrightarrow \text{CONSTRUCTING HOLOMORPHIC } \mathcal{X}_\gamma}$$

THE LEADING, NO QUANTUM CORRECTIONS, APPROXIMATION:

$$\boxed{\mathcal{X}^{sf}_\gamma = \exp\left[\pi R \zeta^{-1} Z_\gamma + i\,\Theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right]}$$

[A. NEITZKE & B. PIOLINE]

- CHECK

$$\varpi = \frac{1}{8\pi^2 R}\,\epsilon_{ij}\,\frac{d\mathcal{X}_i}{\mathcal{X}_i}\wedge\frac{d\mathcal{X}_j}{\mathcal{X}_j}$$

$$= -\frac{i}{2\zeta}\,\omega_+ + \omega_3 - \frac{i}{2}\,\zeta\,\omega_-$$

$$= \frac{1}{8\pi}\left[\frac{i}{\zeta}\langle dZ, d\theta\rangle + \cdots\right]$$

$$\langle dZ, d\theta\rangle = -da^I \wedge dZ_I$$

$\Longrightarrow$ RECOVER g^{sf}

6. SINGLE-PARTICLE CORRECTIONS

NOW WE INCLUDE THE FIRST Q.C.

- FOR SIMPLICITY CONSIDER $r = 1$.

- CONSIDER A POINT $u_* \in \mathcal{M}_V$

WHERE A SINGLE HM HAS $M \to 0$

$\Longrightarrow$ DOMINANT CONTRIBUTION NEAR u_*.

CHOOSE DUALITY FRAME SO IT HAS CHARGE $(q, 0)$, $q > 0$

KK REDUCTION $\Longrightarrow$ TARGET SPACE METRIC IS A GIBBONS-HAWKING ANSATZ: [Seiberg Witten; Ooguri Vafa; Seiberg Shenker]

$$g = V^{-1}(\vec{x}) \left(\frac{d\varphi_m}{2\pi} + A \right)^2 + V(\vec{x}) \, d\vec{x}^2$$

$$F = * dV \qquad \vec{x} \in \mathbb{R}^3$$

HERE:

$$a = x^1 + i x^2$$

$$\varphi_e = 2\pi R\, x^3 \qquad \text{PERIODIC}$$

$$V(\vec{x}) = \frac{g^2 R}{4\pi} \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{g^2R^2|a|^2 + \left(g\frac{\varphi_e}{2\pi} + n\right)^2}}$$

$$= V^{sf} + V^{inst}$$

$$V^{sf} = -\frac{g^2 R}{4\pi}\left(\log\frac{a}{\Lambda} + \log\frac{\bar{a}}{\bar{\Lambda}}\right)$$

$$V^{inst} = \frac{g^2 R}{2\pi} \sum_{n\neq 0} e^{i n g \varphi_e} K_0\left(2\pi R|nga|\right)$$

$$\sim e^{-2\pi R|nga|} \; : \; \text{INSTANTON CONTRIBUTION}$$

NOW, WHAT ARE THE HOLO. FUNCTIONS ON TWISTOR SPACE?

ALGEBRA OF HOLO FUNCTIONS $\{\mathcal{X}_\gamma\}$ ON TWISTOR SPACE IS GENERATED BY:

$$\mathcal{X}_e := \mathcal{X}_{(1,0)} = \exp\{i\varphi_e + \cdots\}$$

$$\mathcal{X}_m := \mathcal{X}_{(0,1)} = \exp\{i\varphi_m + \cdots\}$$

$$\mathcal{X}_{(k_1,k_2)} = (\mathcal{X}_e)^{k_1}(\mathcal{X}_m)^{k_2}$$

$$(k_1,k_2) \in \mathbb{Z}^2 \cong \Gamma_u$$

DETERMINE χ_e AND χ_m

FROM A DIFFERENTIAL EQUATION

HK STRUCTURE: $\alpha = 1, 2, 3$:

$$\omega^\alpha = dx^\alpha \wedge \left(\frac{d\varphi_m}{2\pi} + A \right) + \frac{1}{2} V \epsilon^{\alpha\beta\gamma} dx^\beta dx^\gamma$$

$\Longrightarrow$ COMPUTE $\varpi = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_-$

$$\boxed{\varpi = -\frac{1}{4\pi^2 R} \frac{d\chi_e}{\chi_e} \wedge \frac{d\chi_m}{\chi_m}}$$

WE FIND:

$$\mathcal{X}_e = \mathcal{X}_e^{sf} = \exp\left[\frac{\pi R}{\zeta} a + i \varphi_e + \pi R \zeta \bar{a} \right]$$

<u>**BUT**</u>

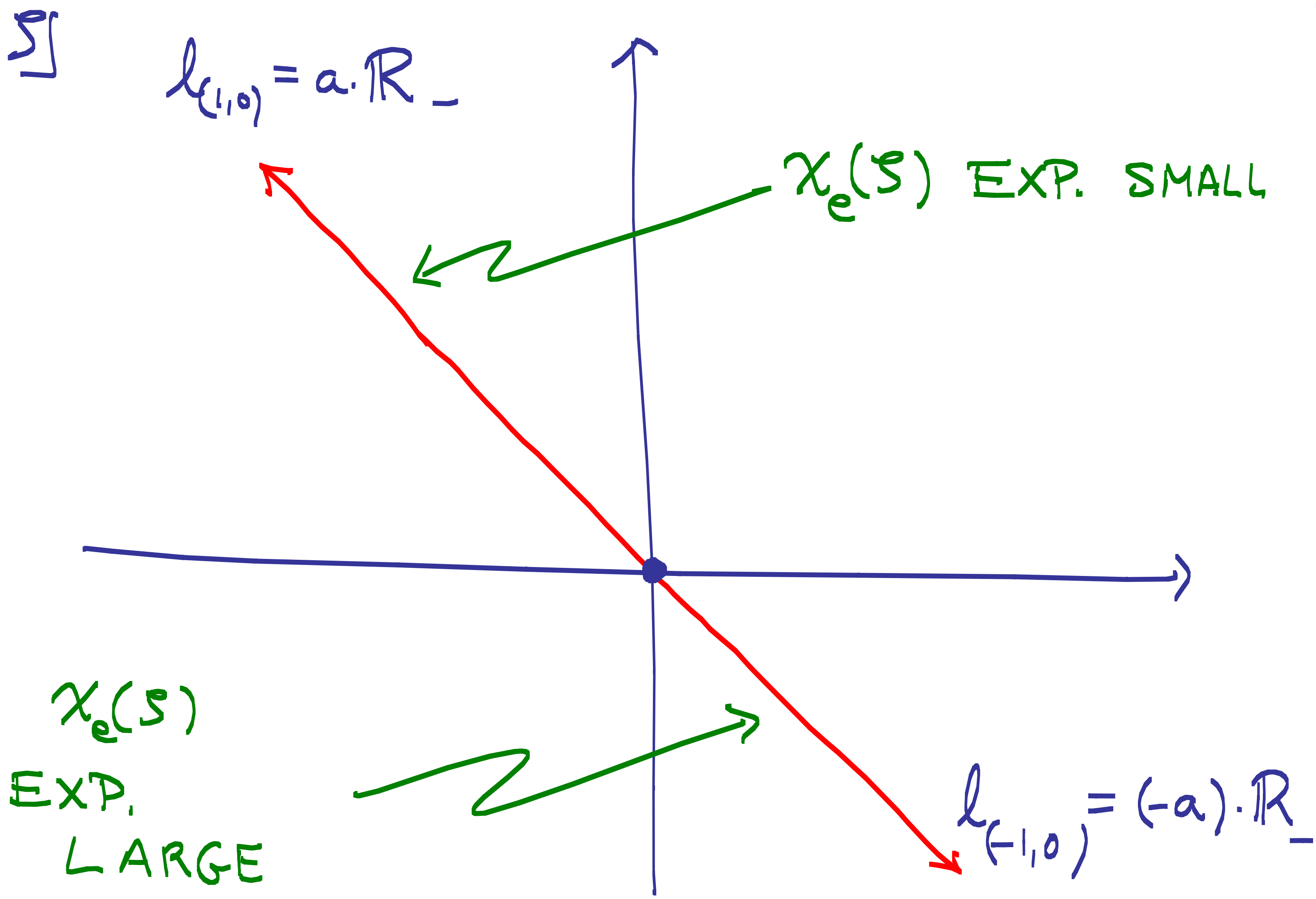
$$\mathcal{X}_m = \mathcal{X}_m^{s.f.} \cdot \mathcal{X}_m^{inst.}$$

$$\mathcal{X}_m^{sf} = \exp\left[\frac{\pi R}{\zeta} \cdot a_D + i \varphi_m + \pi R \zeta \bar{a}_D \right]$$

$$a_D = \frac{q^2}{2\pi i} \left(a \log \frac{a}{e\Lambda} \right)$$

$$\mathcal{X}_m^{inst} = \text{INSTANTON CONTRIBUTION}$$

$$\boxed{\mathcal{X}_m^{\text{inst}}(\zeta) = \exp\left\{ \frac{iq}{4\pi}\int_{\ell_{(1,0)}} \frac{d\zeta'}{\zeta'}\,\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left(1-\mathcal{X}_e(\zeta')^{q}\right) - \frac{iq}{4\pi}\int_{\ell_{(-1,0)}} \frac{d\zeta'}{\zeta'}\,\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left(1-\mathcal{X}_e(\zeta')^{-q}\right)\right\}}$$



KEY FEATURES OF THE SOLUTION

1. AS A FUNCTION OF ζ, $\mathcal{X}_m$ IS DISCONTINUOUS ACROSS THE BPS RAYS:

$$\ell_\gamma := \left\{ \zeta \;\middle|\; \frac{Z_\gamma}{\zeta} \in \mathbb{R}_- \right\}$$

FOR $\gamma = (\pm 1, 0)$, GENERATORS OF Γ_{el}

2. ACROSS THESE RAYS:

$$(\mathcal{X}_e, \mathcal{X}_m)^{cw} = U_\gamma (\mathcal{X}_e, \mathcal{X}_m)^{ccw}$$

$$= \left(\mathcal{X}_e, \mathcal{X}_m (1 - \mathcal{X}_e^{\pm q})^{\mp q}\right)^{ccw}$$

3. $\forall \gamma$, $\mathcal{X}_\gamma \underset{\zeta \to 0, \infty}{\sim} \mathcal{X}_\gamma^{s.f.} (1 + O(1))$

OBSERVATION: X_γ ARE THE SOLUTION OF A RIEMANN-HILBERT PROBLEM.

R-H PROBLEM:

FIND A PIECEWISE HOLO. FUNCTION WITH PRESCRIBED SINGULARITIES AND ASYMPTOTICS.

7. MULTI-PARTICLE CONTRIBUTIONS

TO TAKE INTO ACCOUNT ALL BPS PARTICLES WE CANNOT USE A LOW ENERGY EFFECTIVE LAG., BECAUSE THE PARTICLES WILL BE MUTUALLY NONLOCAL.

PROPOSAL: THE HOLOMORPHIC FUNCTIONS ARE CONSTRUCTED FROM A RIEMANN-HILBERT PROBLEM IN THE ζ- PLANE FOR THE MAPS

$$\mathcal{X}(\zeta) : \mathcal{g} \longrightarrow T = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^*$$

PIECEWISE HOLOMORPHIC IN ζ

RIEMANN-HILBERT PROBLEM:

1.) $\mathcal{X}(\zeta)$ IS DISCONTINUOUS ACROSS BPS RAYS ℓ_γ:

$$\mathcal{X}^{cw} = S_\gamma\left(\mathcal{X}^{ccw}\right)$$

[RECALL: $S_\gamma = \prod_{\ell_{\gamma'} = \ell_\gamma} \mathcal{U}_{\gamma'}^{\Omega(\gamma';u)}$]

2.) $\mathcal{X}(\zeta)$ HAS ASYMPTOTICS FOR $\zeta \to 0, \infty$ GIVEN BY $\mathcal{X}^{sf}(\zeta)$, UP TO $\mathcal{O}(1)$ CORRECTIONS

$$Y := (\mathcal{X}^{sf})^{-1}\mathcal{X} : \mathcal{J} \to \mathcal{J}$$

i.e.

$$Y_0 = \lim_{\zeta \to 0} Y(\zeta) \quad \& \quad Y_\infty = \lim_{\zeta \to \infty} Y(\zeta)$$

EXIST

SOLUTION:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}^{sf}_\gamma(\zeta) \cdot$$

$$\exp\left\{ -\frac{1}{4\pi i} \sum_{\gamma' \in \Gamma} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \cdot \right.$$

$$\left. \cdot \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 - \sigma(\gamma') \mathcal{X}_{\gamma'}(\zeta') \right] \right\}$$

ITERATING THIS EQUATION

(AS A SUM OVER TREES...)

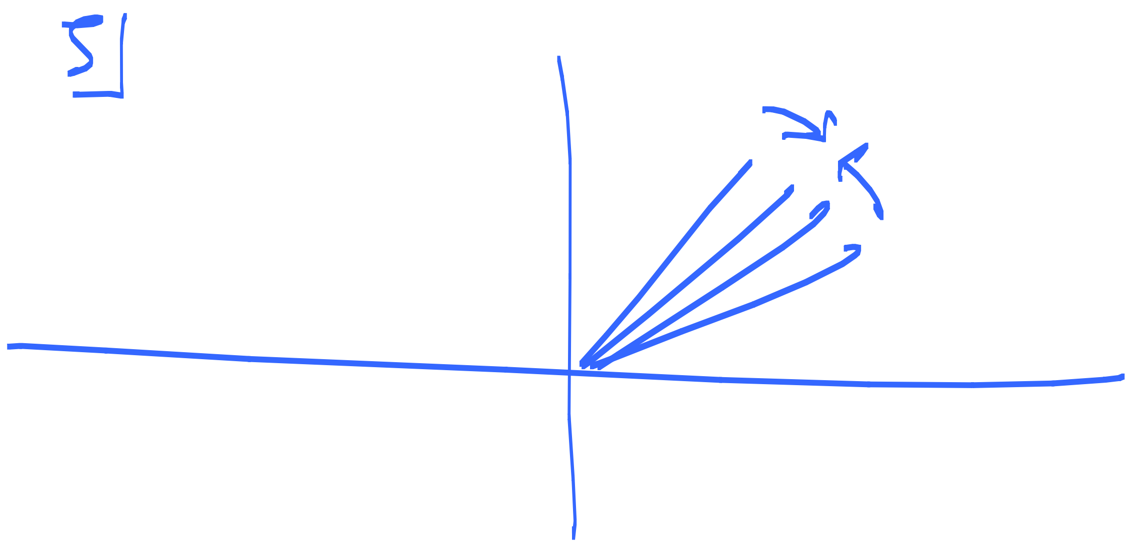
GIVES THE FULL INSTANTON EXPANSION!

$\Rightarrow$ EXPLICIT CONSTRUCTION OF TWISTOR COORD'S

• WE RECONSTRUCT THE METRIC FROM

$$\varpi = \frac{1}{4\pi^2 R} \mathcal{X}^*(\varpi^T)$$

• AS u CROSSES A WALL OF MS BPS RAYS PILE UP



BUT THE JUMP OF $\mathcal{X}$ IN THE RH PROBLEM IS CONTINUOUS AS A FUNCTION OF u: THAT IS THE KS FORMULA

THUS: THE KS FORMULA GUARANTEES THE CONTINUITY OF THE HK. METRIC ACROSS WALLS OF M.S. !

THE RESULTING METRIC PASSES A NUMBER OF CONSISTENCY TESTS.

BUT... WHY IS OUR PROPOSAL THE RIGHT ONE ?

WHY IS THE METRIC THE RIGHT ONE FOR THE PHYSICAL PROBLEM ?

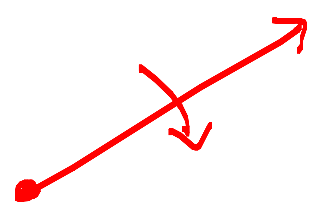
8. PHYSICAL PROOF OF THE KS FORMULA

RH IS EQUIVALENT TO A DIFF. EQ.:

$$\mathcal{A}_\zeta = \mathcal{X}^{-1}\zeta\partial_\zeta \mathcal{X}$$

IS CONTINUOUS IN ζ-PLANE:

ACROSS ℓ_γ



$$\begin{aligned}\mathcal{X}^{-1}\zeta\partial_\zeta\mathcal{X} &\to (S\mathcal{X})^{-1}\zeta\partial_\zeta(S\mathcal{X})\\ &= \mathcal{X}^{-1}\zeta\partial_\zeta\mathcal{X}\end{aligned}$$

$\Rightarrow \mathcal{A}_\zeta$ IS HOLOMORPHIC FOR $\zeta \in \mathbb{C}^*$

$$\Longrightarrow \quad \zeta \partial_\zeta X = A_\zeta X \qquad 2$$

STRUCTURE GROUP: SYMPL(Γ)

ASYMPTOTICS $\Rightarrow$

$$A_\zeta = \zeta^{-1} A_\zeta^{(-1)} + A_\zeta^{(0)} + \zeta A_\zeta^{(+1)}$$

SINCE S_γ IS INDPT. OF R, u, Λ ....

SAME ARGUMENT $\Rightarrow$ X SATISFIES A

SET OF DIFFERENTIAL EQUATIONS:

$$\frac{\partial}{\partial u}\mathcal{X} = \mathcal{A}_u \cdot \mathcal{X}$$

$$\frac{\partial}{\partial \bar{u}}\mathcal{X} = \mathcal{A}_{\bar{u}} \cdot \mathcal{X}$$

$$\Lambda \frac{\partial}{\partial \Lambda}\mathcal{X} = \mathcal{A}_{\Lambda} \cdot \mathcal{X}$$

$$\bar{\Lambda} \frac{\partial}{\partial \bar{\Lambda}}\mathcal{X} = \mathcal{A}_{\bar{\Lambda}} \cdot \mathcal{X}$$

$$R \frac{\partial}{\partial R}\mathcal{X} = \mathcal{A}_R \cdot \mathcal{X}$$

$$\zeta \frac{\partial}{\partial \zeta}\mathcal{X} = \mathcal{A}_{\zeta} \cdot \mathcal{X}$$

$$\mathcal{A}_i = \zeta^{-1} \mathcal{A}_i^{(-1)} + \mathcal{A}_i^{(0)} + \zeta \mathcal{A}_i^{(+1)}$$

KEY POINT: <u>THESE EQUATIONS ALL FOLLOW FROM THE PHYSICS OF THE 4D GAUGE THEORY!!</u>

$$\frac{\partial}{\partial u}\mathcal{X} = \mathcal{A}_u \cdot \mathcal{X}$$
$$\frac{\partial}{\partial \bar{u}}\mathcal{X} = \mathcal{A}_{\bar{u}} \cdot \mathcal{X}$$

HOLOMORPHY ON $\mathcal{M}_g$

ALSO HOLOMORPHY...

$$\Lambda \frac{\partial}{\partial \Lambda}\mathcal{X} = \mathcal{A}_\Lambda \cdot \mathcal{X}$$
$$\bar{\Lambda} \frac{\partial}{\partial \bar{\Lambda}}\mathcal{X} = \mathcal{A}_{\bar{\Lambda}} \cdot \mathcal{X}$$

VIEW Λ AS BACKGROUND VEV OF A VM.

$$R \frac{\partial}{\partial R}\mathcal{X} = \mathcal{A}_R \cdot \mathcal{X}$$
$$\zeta \frac{\partial}{\partial \zeta}\mathcal{X} = \mathcal{A}_\zeta \cdot \mathcal{X}$$

ANOMALOUS SCALE AND R-SYMMETRY

STOKES PHENOMENON

THE ζ- DIFF. EQ. HAS AN IRREGULAR SINGULAR POINT AT $\zeta = 0, \infty$;
SOLUTIONS EXHIBIT STOKES PHENOM.

$\mathcal{A}_\gamma^{(-1)}$ IS CONJUGATE TO $Z \Rightarrow$

- STOKES RAYS = BPS RAYS ℓ_γ

DENOTE STOKES FACTORS BY $\mathcal{S}_\gamma$

REMAINING EQUATIONS:

ISOMONODROMIC DEFORMATION

$\Rightarrow$ STOKES FACTORS $\mathcal{S}_\gamma$ ARE INDP'T OF $R, u, \Lambda, \ldots$

$\Rightarrow$ CHECK AT LARGE R IN 1-INSTANTON APPROXIMATION:

$$\mathcal{S}_\gamma = S_\gamma^{K.S.}$$

9. TAKE-HOME SUMMARY

1. WE CONSTRUCT THE HK METRIC FOR CIRCLE-COMPACTIFICATION OF $\mathcal{N}=2, D=4$ FIELD THEORIES.

2. QUANTUM CORRECTIONS TO THE DIMENSIONAL REDUCTION METRIC COME FROM BPS STATES.

3. CONTINUITY OF THE QUANTUM-CORRECTED METRIC IS EQUIVALENT TO THE KS WCF.

4. USE TWISTOR TRANSFORM AND WRITE HOLOMORPHIC FUNCTIONS AS AN EXPLICIT SUM OVER BPS INSTANTONS.

10. CONCLUSION

— OTHER THINGS WE HAVE DONE —

- MASSIVE HM'S: GENERALIZES KS.

- THERE ARE STRONG CONNECTIONS WITH THE tt^* EQUATIONS OF CECOTTI & VAFA.

- THE FUNCTIONS $\mathcal{X}_\gamma$ ARE 'tHOOFT-WILSON-MALDACENA LOOP OPERATOR VEV'S; MOREOVER THERE IS A NICE INTERPRETATION IN TERMS OF A 3D TFT [TO APPEAR]

- THE MODULI SPACE $(\mathcal{M}, g)$ IS A MODULI SPACE OF A HITCHIN SYSTEM [S. CHERKIS & A. KAPUSTIN]. FOR $SU(2)$ WE HAVE CONSTRUCTED IT AS A SPACE OF STOKES MATRICES GLUED BY KS TRANSFORMATIONS. [TO APPEAR.]

— TO DO —

- UNDERSTAND BETTER THE NEED FOR THE QUADRATIC REFINEMENT.
- SINGULARITIES AT SUPERCONFORMAL POINTS REMAIN TO BE UNDERSTOOD
- RELATIONS TO INTEGRABLE SYSTEMS
- RELATION TO THE WORK OF JOYCE & BRIDGELAND / TOLEDANO LAREDO
- SOME OF THE DISCUSSION GENERALIZES NICELY TO SUGRA, BUT PUZZLES REMAIN
- MIGHT GIVE EXPLICIT FORMULATION OF Q.C.'S TO HYPERMULTIPLET MODULI SPACES.