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Trieste Lectures on Mathematical Aspects of Supersymmetric Black Holes

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ABSTRACT: Outline of some lectures: "Mathematical Aspects of $\mathcal{N} = 2$ BPS black holes" THESE ARE VERY PRELIMINARY. VERY ROUGH IN SOME PLACES, AND TO BE USED ONLY AT THIS SCHOOL! Comments on the current draft are welcome.

Contents

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1. Introduction and Overview of the Lectures

This is a series of lectures about BPS states in string theory and in particular how to count them. The main motivation for this work is the program, initiated by Strominger and Vafa in 1995, of trying to account for the entropy of supersymmetric black holes in terms of the microstates described by D-branes. We recall some of the features of the Strominger-Vafa computation in section 1.1 below.

An important role in BPS statecounting has been played by automorphic functions, and in particular by modular forms for $SL(2,\mathbb{Z})$, so our first lecture is devoted to basic aspects of the role of modular forms in two-dimensional conformal field theory. We will emphasize the interplay between holomorphy, modularity, and the important role of "polar terms" in modular forms of negative weight.

In the second lecture we combine these ideas with extended supersymmetry. In particular, for $\mathcal{N}=(2,2)$ superconformal theories the elliptic genus is a holomorphic modular object - a weak Jacobi form of weight zero (see definitions in section **** below). We show how one can construct the entire elliptic genus given just the degeneracies of the "polar states" through a Poincaré series (aka the Fareytail expansion). A corollary of this construction is the Rademacher formula for nonpolar degeneracies in terms of the polar degeneracies.

Then, using the recent activity in 2+1 dimensional quantum gravity we motivate the consideration of extremal conformal field theories. In particular, we describe a very new result, obtained in [15] concerning the possible existence of extremal $\mathcal{N}=(2,2)$ conformal field theories.

Given the great success of the Strominger-Vafa computation, an obvious program is to repeat the story for more realistic black holes. That is, we would like to carry out a similar computation for black holes in four dimensions with fewer unbroken supersymmetries. The state of the art in this program is that in four dimensions with $\mathcal{N}=8$ unbroken supersymmetries we have very good control. This is the subject of Ashoke Sen's lectures. In the case when there are $\mathcal{N}=4$ unbroken supersymmetries we know some things, but much there is a much less complete picture. In particular, we do not even know how to compute microscropic entropies in certain natural charge regions, for example, when all charges are uniformly scaled to infinity. The subject of the final lectures concerns the behavior of the index of BPS states for BPS states associated to D-branes on Calabi-Yau manifolds. We stress the fact that the index is not constant, but has wall-crossing behavior.

1.1 The Strominger-Vafa computation

The main motivation for reviewing the old results on modular invariance and the elliptic genus is that they have applications to the ongoing Strominger-Vafa program of accounting for the Beckenstein-Hawking entropy of black holes in terms of counting of microstates.

Strominger and Vafa initiated this program in [35] in the context of 5-dimensional supersymmetric black holes. The microstate counting was provided by the theory of D-branes. Let us briefly recall the most important points. Reviews include [23, 29, 8, 7, 30].

Strominger and Vafa considered type IIB string theory on $\mathbb{R}^{1,4}\times S^1\times K3$ and considered a system of Q_1 $D1$ branes wrapping the S^1 and Q_5 $D5$ branes wrapping $S^1\times K3$. They considered the case in which the radius of S^1, denoted R is large in string units and argued that the low energy states in this system are described by a superconformal field theory on $S^1\times\mathbb{R}$ with target space $\mathrm{Sym}^{Q_1Q_5}(K3)$. [1] This superconformal field theory has $(4,4)$ supersymmetry and central charge $c=\tilde{c}=6Q_1Q_5$. The elliptic genus counting the index of BPS states

$$\chi(\tau,z;\mathrm{Sym}^N(K3))=\sum_{n,\ell}c^{(N)}(n,\ell)q^ny^\ell \tag{1.1}$$

has been accounted for above.

On the other hand, there is a spacetime supergravity point of view. The quantum numbers Q_1,Q_5 specify RR charges and n a KK momentum charge. For large charges, the BPS states are described semiclassically by a unique black hole which, in (4+1)-dimensional Einstein frame has the metric

$$ds^2=-(f_1f_5f)^{-2/3}dt^2+(f_1f_5f)^{1/3}ds^2_{\mathbb{R}^4} \tag{1.2}$$

$$\begin{aligned} f_1&=1+\frac{4GR}{g_s\pi\alpha'}\frac{Q_1}{r^2}\\ f_5&=1+\alpha' g_s\frac{Q_5}{r^2}\\ f&=1+\frac{4G}{\pi R}\frac{n}{r^2}\end{aligned} \tag{1.3}$$

♣ NEED TO CHECK THIS! ♣ From (1.2) one computes the Beckenstein-Hawking entropy

$$S_{BH}=2\pi\sqrt{Q_1Q_5n} \tag{1.4}$$

On the other hand, we know that for $n\gg Q_1Q_5$ we have the asymptotics

$$\sum_\ell c^{(Q_1Q_5)}(n,\ell)\sim e^{2\pi\sqrt{Q_1Q_5n}} \tag{1.5}$$

[1] This is, roughly speaking, the symmetric product $(K3)^{Q_1Q_5}/S_{Q_1Q_5}$. To be more precise, this orbifold (which has singularities) has a resolution of singularities called the Hilbert scheme of points $Hilb^{Q_1Q_5}(K3)$, a smooth algebraic variety. It inherits a hyperkahler metric from that on $K3$. This Hyperkahler resolution is the true target space of the conformal field theory. See [28] for a description of the relevant mathematics and [REFS] for some description of how one arrives at this conclusion.

giving perfect agreement.

♣ MORE Precise? Add spin? ♣

Strictly speaking, for this successful computation one only needs the Cardy formula. However, the techniques we have described become much more relevant when we go on to the next steps in the SV program, and attempt to give a microscopic account of the entropy of four-dimensional black holes preserving only 4 supersymmetries.

Remarks

1. ♣ Explain the link to the AdS/CFT correspondence? ♣
2. ♣ Explain that the entropy is dominated by the long string with $c_{eff} = Q_1 Q_5$? (Maybe application of the symmetric product section) ♣

2. Modularity in 2D Conformal Field Theory

2.1 Introduction

Modular forms and automorphic functions have been playing an important role in mathematics since the early 19th century, and continue to be an active and fascinating subject of research to this day. The theory of modular forms entered physics in the 1970's and 1980's in the context of string theory and 2-dimensional conformal field theory. In 2D conformal field theory, modular invariance puts strong constraints on the spectrum of the theory [Cardy]. In string theory, modular invariance is part of the anomaly-cancellation and consistency conditions for a string theory.

2.2 Partition functions of 2D conformal field theory on a torus

Suppose we have a 2D conformal field theory $\mathcal{C}$ with Hilbert space $\mathcal{H}$. It is a representation of left- and right-moving Virasoro algebras with central charges $c, \tilde{c}$:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{2.1}$$

and similarly for $\tilde{L}_n$.

We will assume that $\mathcal{H}$ can be decomposed into a (possibly infinite) direct sum of highest weight representations V_h. Recall these are representations built on a vacuum vector $|h\rangle$ with

$$L_0|h\rangle = h|h\rangle \qquad L_n|h\rangle = 0, \qquad n > 0 \tag{2.2}$$

In particular, the spectrum will be assumed discrete.

One of the most useful quantities we can associate to it is the partition function. We define $q := e^{2\pi i\tau}$, with

$$\tau = \theta + i\beta.$$

A common notation in the math literature is $e(x) := e^{2\pi i x}$, so we could write $q = e(\tau)$. Then the partition function is defined to be:

$$Z(\tau, \bar{\tau}) := \mathrm{Tr}_{\mathcal{H}} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - c/24} \tag{2.3}$$

$$\tag{2.4}$$

If the spectrum of H on $\mathcal{H}$ is bounded below and discrete then $Z(\tau, \bar{\tau})$ is real analytic for τ in the upper half-plane $\mathbb{H}$, although it might have singularities for $\mathrm{Im}\tau \to \infty$ or $\mathrm{Im}\tau \to 0$. [2]

The partition function (2.3) has the interpretation of being the path integral on a flat torus with modular parameter τ. To see this we write

$$Z(\tau, \bar{\tau}) = \mathrm{Tr}_{\mathcal{H}} e^{-2\pi\beta H + 2\pi i\theta P} \tag{2.5}$$

where the Hamiltonian is

$$H = L_0 + \tilde{L}_0 - (c + \tilde{c})/24$$

while the momentum is:

$$P = L_0 - \tilde{L}_0 - (c - \tilde{c})/24$$

Thus, we propagate for Euclidean time β and then glue-via the trace- with a twist θ as shown in (1).

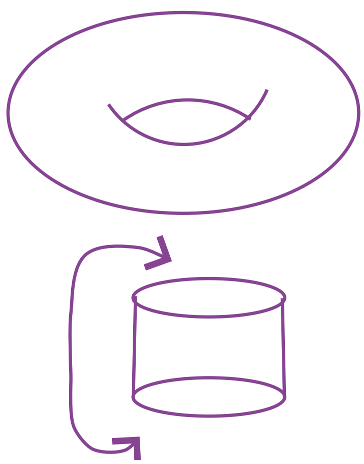

Figure 1: Taking the trace after propagating the closed string is the path integral on a torus.

The importance of this observation is that we can now study the behavior of the theory under diffeomorphisms of the torus. The group of topologically nontrivial orientation preserving diffeomorphisms of the torus is $SL(2, \mathbb{Z})$, and representatives are easily written.

It is useful to transform to coordinates so that

$$\xi = s + it = \sigma^1 + \tau\sigma^2 \tag{2.6}$$

Here s is the spatial and t the time coordinate. If we impose the identifications $\sigma^1 \sim \sigma^1 + 1$ then space is identified with period 1. If we furthermore impose $\sigma^2 \sim \sigma^2 + 1$ then t is identified with period β together with a twist by θ in s. In these coordinates we identify the torus as

$$E_\tau := \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}). \tag{2.7}$$

[2]♣ MENTION THAT SL2R gives counterexample when spectrum is not discrete - due to noncompactness of the target worldvolume ♣

as shown in figure ***.

Let us now consider the diffeomorphisms acting on (σ^1, σ^2) as:

$$(\sigma^1, \sigma^2) \rightarrow (d\sigma^1 + b\sigma^2, c\sigma^1 + a\sigma^2)$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{2.8}$$

We denote $\Gamma = SL(2, \mathbb{Z})$. It is called the *modular group.*

The action of the modular group on z is:

$$\xi \rightarrow (c\tau + d)(\sigma^1 + \frac{a\tau + b}{c\tau + d}\sigma^2)$$

and up to an overall scaling this induces a fractional linear transformation on τ:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \tag{2.9}$$

Note that the action factors through an action of $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$.

Put differently: If we view the complex structure of the torus τ as induced from the Riemannian metric $ds^2 = |d\xi|^2$ then the pullback metric is

$$f^*(ds^2) = |c\tau + d|^2 \left| d\sigma^1 + \frac{a\tau + b}{c\tau + d} d\sigma^2 \right|^2$$

so we have a Weyl scaling and a modular transformation on τ.

Now, in a diffeomorphism-invariant theory the partition function Z must be diffeomorphism invariant. The effect of a Weyl rescaling on the background metric $\hat{g}$ on a Riemann surface Σ, $\hat{g} \rightarrow e^{\phi}\hat{g}$ on a partition function is

$$Z \rightarrow \exp \left[const.(c + \tilde{c}) \int_{\Sigma} \sqrt{\hat{g}} \mathcal{R}(\hat{g}) \phi \right] Z \tag{2.10}$$

but this is zero for $\Sigma = E_\tau$ because the background metric is flat (more generally, ϕ is constant and the Euler character of E_τ is zero). Therefore, the partition function must be modular invariant. Thus, if we know a theory to be diffeomorphism invariant, modular invariance puts a nontrivial constraint on the spectrum of the theory. Conversely, if we are given a theory, we can use modular transformations of its partition function as a diagnostic to search for anomalies under globally nontrivial diffeomorphisms.

It thus behooves us to review a few:

Facts about the modular group

1. The center is $\{\pm 1\}$. The action of Γ factors through an action of $\bar{\Gamma} = PSL(2, \mathbb{Z})$ on τ. That is $\gamma = -1$ acts trivially on τ and only $\bar{\Gamma}$ acts effectively.

2. The modular group is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.11}$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.12}$$

with relations $S^2 = -1$ and $(ST)^3 = -1$ (which implies $(TS)^3 = -1$). Moreover, the only torsion elements in Γ have orders $2, 3, 4, 6$. If γ has order 2 it is -1. If it has order 4 it is conjugate to $\pm S$. If it has order 3 it is conjugate to $-(ST)^{\pm 1}$ and if it has order 6 it is conjugate to $(ST)^{\pm 1}$.

Group-theoretically, $PSL(2,\mathbb{Z})$ is the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ while $SL(2,\mathbb{Z})$ is the amalgamated product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. An amalgamated product $G_1 *_A G_2$ relative to homomorphisms $f_i : A \to G_i$ is the quotient of the free product by the relations $ag_i^{-1} = 1$ if $f_i(a) = g_i$. See [33] for more information.

3. There is an algorithm to express a general element $\gamma \in SL(2,\mathbb{Z})$ in terms of a word in S, T using the continued fraction expansion of ratios of the matrix elements of γ.

4. The modular images of $\tau = i\infty$ are the rational numbers: $\gamma(\imath\infty) = \frac{a}{c} \in \mathbb{Q}$.

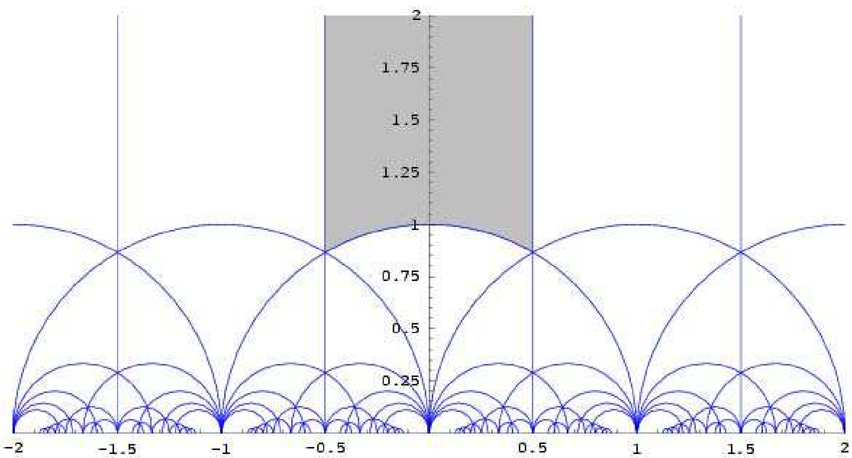

Figure 2: A standard choice of fundamental domain for the action of $PSL(2,\mathbb{Z})$ on the upper half-plane. The orbifold points are at $\tau = i$ and $\tau = e^{i\pi/3} \sim e^{2\pi i/3}$ and their images. The blue triangles are all modular images of the fundamental domain $\mathcal{F}$. This picture was lifted from Wikipedia.

5. A standard fundamental domain for the action of Γ on $\mathbb{H}$ is the keyhole region $\mathcal{F}$ shown in figure 2. $\mathbb{H}/\Gamma$ has a $\mathbb{Z}_2$ orbifold singularity at $\tau = i$ since $S^2 = 1$ in $PSL(2,\mathbb{Z})$ and a $\mathbb{Z}_3$ singularity at $\tau = e^{i\pi/3}$ since $(ST)^3 = 1$ in $PSL(2,\mathbb{Z})$. The cusp at $\tau = i\infty$ is preserved by the subgroup generated by $T : \tau \to \tau + 1$, i.e.

$$\Gamma_\infty = \{ \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \}_{\ell \in \mathbb{Z}} \tag{2.13}$$

6. Let $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{i\infty\}$ then $(\mathbb{H} \cup \hat{\mathbb{Q}})/\Gamma$ can be given the structure of an analytic Riemann surface, and is therefore $\mathbb{C}P^1$. Therefore there is a 1-1 uniformizing map denoted $j(\tau)$, and defined up to a constant if j takes ∞ to ∞. Therefore, the field of meromorphic functions invariant under Γ is $\mathbb{C}(j)$.

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SAY SOMETHING ABOUT HAUPTMODUL

NEED TO STRESS MORE THAT YOU ARE TALKING ABOUT UNITARY THEORIES

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2.3 Chiral splitting and holomorphy

If all we know is that $Z(\tau, \bar{\tau})$ is singularity free, real analytic and modular invariant we cannot conclude very much: Take any function with compact support in $\mathcal{F}$ and average it over the modular group. The result is such a function. However, modular invariance places strong constraints when combined with holomorphy in τ.

Because the conformal field theory splits locally into left- and right-moving degrees of freedom one can, in general decompose the partition function as a sum of the form

$$Z(\tau, \bar{\tau}) = \sum_i f_i(\tau) \tilde{f}_i(\bar{\tau}) \tag{2.14}$$

where $f_i, \tilde{f}_i$ are holomorphic functions.

For example, we can always decompose into characters of the Virasoro highest weight representations:

$$Z(\tau, \bar{\tau}) = \sum_{h,\tilde{h}} N_{h,\tilde{h}} f_h(\tau) \overline{\tilde{f}_{\tilde{h}}(\tau)} \tag{2.15}$$

where f_h has the expansion

$$f_h(\tau) = \sum_{n \geq 0} \hat{f}_h(n) e(\tau(n - \Delta_h)) \tag{2.16}$$

with $\Delta_h = \frac{c}{24} - h$.

But we can also extend the Virasoro algebra in various ways to get more control of the spectrum. In particular, we will do this with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetry.

One finds especially strong constraints in the case that the sum in (2.14) is a finite sum. This happens in a broad class of theories known as *rational conformal field theories*. Evidently, the invariance of Z means that f_i must then transform under Γ as:

$$f_i(\gamma(\tau)) = M_{ij}(\gamma) f_j(\tau) \tag{2.17}$$

where $M(\gamma)$ is a projective representation of the the modular group. Note that $\tilde{f}_i$ transforms in the contragredient representation.

Remarks

1. The most extreme case of all is when there is one term and the partition function is holomorphic $Z = f(\tau)$. $f(\tau)$ is a then a Γ-invariant function. Such a function is known as a *modular function.* As we have explained above, the field of functions meromorphic on $\mathbb{H}$ which are invariant under Γ is the field of rational expressions in the famous modular j function. We will show later how to compute its q-expansion:

$$j(\tau) = q^{-1} + 196884q + 21493760q^2 + \dots \tag{2.18}$$

   $j(\tau)$ can have zeroes, but does not have any poles in $\mathbb{H}$. Note that there is a singularity in j for $\tau \to i\infty$.

   Since our $Z(\tau)$ has a spectrum bounded below and no other singularities in $\mathbb{H}$ it must follow that $Z(\tau)$ is a polynomial in $j(\tau)$.

2. In general, if

$$f_i(\tau) = \sum_{n\geq 0} \hat{f}_i(n) e^{2\pi i\tau(n-\Delta_i)} \tag{2.19}$$

   where $\Delta_i = \frac{c}{24} - h_i$. Based on this expansion we divide the states in the representation space into two kinds:

   - *Polar states* have $n - \Delta_i < 0$. They contribute a singularity to $f_i(\tau)$ in the limit $\mathrm{Im}\tau \to \infty$, or, equivalently, $q \to 0$.
   - Nonpolar states have $n - \Delta_i > 0$

   states with $n - \Delta_i = 0$ are more subtle and should be considered polar or nonpolar depending on the example.

3. To illustrate the power of holomorphy plus modularity, we will demonstrate in the present example of a holomorphic partition function how the *finite* set of polar degeneracies determine the *infinite* set of nonpolar degeneracies. We will discuss this in much more detail, but let us consider the example $Z(\tau,\bar{\tau}) = f(\tau)$, and $M(\gamma) = 1$ is a 1×1 matrix. We may expand

$$f(\tau) = \hat{f}_0 e(-\Delta\tau) + \hat{f}_1 e((1-\Delta)\tau) + \cdots \tag{2.20}$$

   with Δ some integer. But, as we have said we also know that

$$f(\tau) = a_0 j^{\Delta} + a_1 j^{\Delta-1} + \cdots + a_{\Delta} j^0 \tag{2.21}$$

   Since the partition function must be nonsingular for $\tau \in \mathbb{H}$ Δ will be a nonnegative integer. Moreover, for the same reason the series must terminate at order j^0. Now, equating the polar terms in these two expressions gives a triangular system of equations for the a_i in terms of the polar degeneracies $\hat{f}_j$, $j = 0, \dots, \Delta$. Thus we can solve for the a_i in terms of the polar degeneracies. The partition function is then completely fixed! That is, there is no $+\cdots$ in (2.21). Therefore, all the higher degeneracies are captured by the polar degeneracies $\hat{f}_j$, $j = 0, \dots, \Delta$.

4. There are many examples where the sum in (2.14) is infinite and yet one can still obtain (2.17), but in general when the sum is infinite this cannot be done.

2.4 A basic example: Periodic chiral scalars

2.4.1 The Gaussian model

A simple instructive example is the Gaussian model of a single real scalar field X, with $X \sim X + 1$ and

$$S = \frac{(2\pi R)^2}{4\pi\alpha'} \int dX * dX \tag{2.22}$$

A standard computation in CFT leads to:

$$Z = \frac{\Theta_{\Lambda_R}}{\eta\bar{\eta}} \tag{2.23}$$

Here η is the Dedekind function arises from quantizing the oscillators:

$$\eta(\tau) := e^{\frac{2\pi i\tau}{24}} \prod_{n=1}^{\infty}(1 - q^n) \tag{2.24}$$

To define the numerator, first, Θ_Λ is an example of a *Siegel-Narain theta function*. Let $\mathbb{R}^{1,1}$ be a Euclidean space of signature $(1,1)$. Vectors have left- and right-moving projections $v = (v_+; v_-)$ with norm $v^2 = v_+^2 - v_-^2$. If $\Lambda \subset \mathbb{R}^{1,1}$ we define

$$\Theta_\Lambda := \sum_{p\in\Lambda} q^{\frac{1}{2}p_+^2} \bar{q}^{\frac{1}{2}p_-^2} \tag{2.25}$$

The lattice Λ_R is defined by :[3]

$$p = ne + mf \qquad n, m \in \mathbb{Z} \tag{2.26}$$

with

$$e = \frac{1}{\sqrt{2}}(\frac{1}{R}; \frac{1}{R}) \tag{2.27}$$

$$f = \frac{1}{\sqrt{2}}(R; -R) \tag{2.28}$$

As an abstract lattice the e, f span a hyperbolic lattice $II^{1,1}$ since $e^2 = f^2 = 0$ and $e \cdot f = 1$. The embedding of this unique even unimodular lattice of signature $(1,1)$ into $\mathbb{R}^{1,1}$ encodes the radius. Thus

$$p_+ = \frac{1}{\sqrt{2}}(\frac{n}{R} + mR) \tag{2.29}$$

$$p_- = \frac{1}{\sqrt{2}}(\frac{n}{R} - mR) \tag{2.30}$$

$p_\pm$ are often denoted p_L, p_R in the literature. n is interpreted as a momentum eigenvalue and m as a winding eigenvalue.

There are many useful lessons one can extract from this simple example among them are:

[3] We are quantizing on a circle of length 2π, and putting $\hbar = 1$ and $\alpha' = 1$. We can restore α' since it has units of length-squared, while R has units of length.

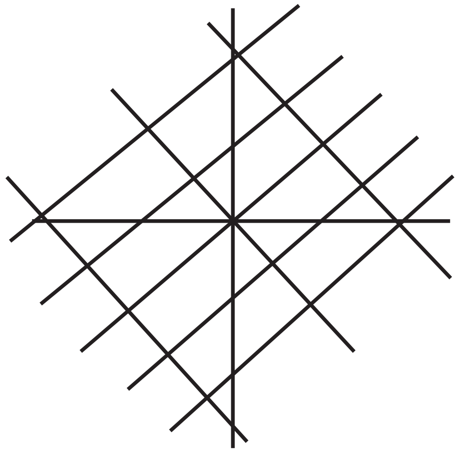

Figure 3: A Narain lattice for the Gaussian model.

1. The Gaussian model clearly has no diffeomorphism anomalies. Therefore $Z(\tau,\bar{\tau})$ is modular invariant. From this we can conclude that the and denominator of (2.23) transform nicely separately under modular transformations. Using the Poisson summation formula one can directly check that
$$\Theta_{\Lambda_R} \to (c\tau+d)^{1/2}(c\bar{\tau}+d)^{1/2}\Theta_{\Lambda_R} \tag{2.31}$$
It follows that
$$\eta(\frac{a\tau+b}{c\tau+d}) = (-i(c\tau+d))^{1/2}e^{i\phi(\gamma)}\eta(\tau) \tag{2.32}$$
where $e^{i\phi(\gamma)}$ is a phase. In fact, one can show that the Dedekind function transforms as:
$$\eta(\tau+1) = e^{\frac{2\pi i}{24}}\eta(\tau) \tag{2.33}$$
$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \tag{2.34}$$
and hence $e^{i\phi(\gamma)}$ is an interesting 24^{th} root of unity. ♣ GIVE IT? ♣

2. The model enjoys T-duality: The theories at radius R and R_D are isomorphic if $RR_D = 1$. One manifestation of this is the invariance of the above partition function under $R \to 1/R$. Note that momentum and winding are exchanged.

3. The $R \to \infty$ limit is interesting. The sum on m becomes suppressed and only the $m = 0$ term survives. The sum on n becomes (recall that $\beta = \mathrm{Im}\tau$)
$$\sum_{n\in\mathbb{Z}} e^{-\pi\beta\left(\frac{n}{R}\right)^2} \to \frac{R}{\beta^{1/2}}\left(1+\mathcal{O}(e^{-\frac{\pi R^2}{\beta}})\right) = \int (2\pi R)\frac{dp}{2\pi}e^{-2\pi\beta\frac{1}{2}p^2}\left(1+\mathcal{O}(e^{-\frac{\pi R^2}{\beta}})\right) \tag{2.35}$$
and hence
$$Z(\tau,\bar{\tau}) \to \frac{R}{(\mathrm{Im}\tau)^{1/2}}\frac{1}{\eta\bar{\eta}}. \tag{2.36}$$
The factor of R is the volume of the target spacetime. Note that we have lost holomorphic factorization.

4. The partition function of a noncompact self-dual or chiral scalar is taken to be

$$Z(\tau) = \frac{1}{\eta} \tag{2.37}$$

Note that it is singular at $\tau \to i\infty$, and has gravitational anomalies. That is, it is a section of a nontrivial line bundle on (a finite cover of) $\mathbb{H}/\Gamma$.

5. When R^2 is rational, $R^2 = p/q$ is in lowest terms, (2.14) becomes a finite sum:

$$Z = \sum_{\mu,\nu=1}^{2pq} f_\mu(\tau) N_{\mu\nu} \overline{f_\nu(\tau)} \tag{2.38}$$

where

$$f_\mu = \frac{\Theta_{\mu,pq}(0,\tau)}{\eta} \qquad \mu = 1, \dots, 2pq \tag{2.39}$$

are expressed in terms of holomorphic level pq theta functions (See equation (3.45) below.) The extended algebra is an extension of the loop group $LU(1)$. The loop group is best thought of in this context as the differential cohomology group $\check{H}^1(S^1)$ (because our remarks will generalize to higher self-dual forms). The extended algebra is characterized by an integrally quantized *level*, and the level is $2pq$. In particular it is never equal to the basic extension at level 1. But it is the level one central extension which gives the basic self-dual scalar field. To obtain the basic self-dual field one must take a double cover of the target space circle at the free-fermion radius $R^2 = 2$. The partition functions of the self-dual field are then given by level *one-half* theta functions with characteristics:

$$Z_\epsilon = \frac{\vartheta\,[\epsilon]\,(0|\tau)}{\eta} \tag{2.40}$$

where ϵ encodes a spin structure on the torus. For further discussion of these points see [14, 13, 5].

6. Remarks 4 and 5 generalize to other important theories of self-dual fields, including the M-theory 5-brane partition function and the partition function of the RR fields in type II string theory. But this is a topic for another lecture series.

2.4.2 General theories of self-dual and anti-self-dual scalars

More generally, if we have b_+ compact left-moving (self-dual) scalars and b_- compact right-moving (anti-self-dual) scalars their partition function is of the form

$$Z(\tau,\bar\tau) = \frac{1}{\eta(\tau)^{b_+}\overline{\eta(\tau)}^{b_-}} \Theta_\Lambda(\tau,\bar\tau) \tag{2.41}$$

Here Θ_Λ is a *Siegel-Narain theta function.* To define it we embed a lattice Λ of signature (b_+, b_-) into $\mathbb{R}^{b_+,b_-}$ with projections $p \to (p_+; p_-)$ onto the positive definite and negative definite subspaces of dimensions $b_\pm$ respectively. That is $\Lambda \otimes \mathbb{R} \cong \mathbb{R}^{b_+,0} \oplus \mathbb{R}^{0,b_-}$. Then

$$\Theta_\Lambda(\tau,\bar\tau) = \sum_{p\in\Lambda} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \tag{2.42}$$

The embedding of Λ in $\mathbb{R}^{b_+,b_-}$ encodes the data of a flat metric, a flat B-field, and a flat gauge field (coupling to the left- and right-moving currents) on the target space torus [16].

To express the transformation properties it is useful to generalize a little bit and define the general Siegel-Narain theta function

$$\Theta_\Lambda(\tau,\bar{\tau};\alpha,\beta;\xi) := \exp[\frac{\pi}{2\beta}(\xi_+^2-\xi_-^2)]\sum_{\lambda\in\Lambda}\exp\left\{i\pi\tau(\lambda+\beta)_+^2+i\pi\bar{\tau}(\lambda+\beta)_-^2+2\pi i(\lambda+\beta,\xi)-2\pi i(\lambda+\frac{1}{2}\beta,\alpha)\right\} \tag{2.43}$$

where $\mathrm{Im}\tau = \beta$.

The main transformation law is:

$$\Theta_\Lambda(-1/\tau,-1/\bar{\tau};\alpha,\beta;\frac{\xi_+}{\tau}+\frac{\xi_-}{\bar{\tau}}) = \sqrt{\frac{1}{|\mathcal{D}|}}(-i\tau)^{b_+/2}(i\bar{\tau})^{b_-/2}\Theta_{\Lambda^*}(\tau,\bar{\tau};\beta,-\alpha;\xi) \tag{2.44}$$

where $\Lambda*$ is the dual lattice, and $\mathcal{D} = \Lambda^*/\Lambda$ is a finite abelian group known as the *discriminant group*. Equation (2.44) can be proven straightforwardly by using the Poisson summation formula.

To get the transformation law under T we must assume that Λ has a *characteristic vector*, that is, a vector w_2, such that

$$(\lambda,\lambda) = (\lambda,w_2) \bmod 2 \tag{2.45}$$

for all λ. In this case we have in addition:

$$\Theta_\Lambda(\tau+1,\bar{\tau}+1;\alpha,\beta;\xi) = e^{-i\pi(\beta,w_2)/2}\Theta_\Lambda(\tau,\bar{\tau};\alpha-\beta-\frac{1}{2}w_2,\beta;\xi) \tag{2.46}$$

Remarks

1. The global gravitational anomalies cancel when Λ is an even unimodular lattice and $b_+ - b_- = 0 \mathrm{mod} 24$. The case $b_- = 0$ gives an example of a conformal field theory with a purely holomorphic partition function.
2. The case of Λ even unimodular occurs in toroidal compactifications of the heterotic string.
3. More importantly for our present theme, these theories, where Λ is *not* even unimodular, arise in the reduction of the M5 brane theory on complex surfaces.

2.5 Vector-valued nearly holomorphic modular forms

The above example motivates the consideration of a class of functions more general than just modular invariant functions. We are interested in vectors of holomorphic functions transforming in some matrix representation of Γ.

For

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

it is useful to define

$$j(\gamma,\tau) := c\tau + d \tag{2.47}$$

Definition: A vector-valued nearly-holomorphic[4] modular form of weight w and multiplier system M is a collection of functions $f_\mu(\tau)$, holomorphic for $\tau \in \mathbb{H}$ such that for all $\gamma \in SL(2,\mathbb{Z})$:

$$f_\mu(\gamma(\tau)) = j(\gamma,\tau)^w M(\gamma)_{\mu\nu} f_\nu(\tau), \tag{2.48}$$

for a matrix $M(\gamma)$ constant in τ.

If w is not an integer we must choose a branch of the logarithm. We choose $-\pi < \arg(z) \leq \pi$. Using the important cocycle identity:

$$j(\gamma_1\gamma_2,\tau) = j(\gamma_1,\gamma_2\tau)j(\gamma_2,\tau), \tag{2.49}$$

one easily proves that the multiplier system $\gamma \to M(\gamma)$ defines a representation of Γ when w is integral, and a projective representation of Γ when w is nonintegral.

It is useful to define the slash notation:

$$(f|\gamma)(\tau) := j(\gamma,\tau)^{-w} M(\gamma)^{-1} f(\gamma\tau) \tag{2.50}$$

which can be applied to any vector of functions. Vector valued nearly holomorphic forms satisfy $f|\gamma = f$.

The simplest case is when the representation of the modular group is trivial. In this case it immediately follows that if f is nonzero then w must be an even integer, as one sees by considering $\gamma = -1$. Note that from the invariance under T we learn that $f(\tau)$ must have a Fourier series expansion:

$$f(\tau) = \sum_{n\in\mathbb{Z}} \hat{f}(n) q^n \tag{2.51}$$

We will see that many interesting physical questions are related to the asymptotic behavior of the Fourier coefficients of modular forms.

When $M(\gamma)$ is the trivial one-dimensional representation then we can derive a useful constraint on any nonzero meromorphic function f with $f|\gamma = f$. Let $v_p(f)$ denote the order of the zero (or pole) of f at $\tau = p \in H$, that is $v_p(f)$ is the integer n such that $f(\tau)/(\tau - p)^n$ is holomorphic and nonzero at $\tau = p$. It is positive if f has a zero at p and negative if f has a pole.

By carefully integrating the one-form $\frac{1}{2\pi i}\frac{df}{f}$ around the boundary of the fundamental domain one derives the constraint:[5]

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum_{p\in\mathbb{H}/\Gamma}^{*} v_p(f) = \frac{w}{12} \tag{2.52}$$

where the sum on the LHS is over the points in the fundamental domain omitting $\tau = i, \rho$. Only a finite number of terms are nonzero.

Remarks

[4]the term "weakly holomorphic" is also used in the literature

[5]For the details see [32], p. 85 or [22], p. 6.

1. In all physics applications I know the projective representation M factors through a a *congruence subgroup*. To define this let $\Gamma(N)$ be the normal subgroup of Γ defined by the kernel of the map $\gamma \to \gamma \mathrm{mod} N$. A congruence subgroup (of level N) is a subgroup Γ' of Γ which contains $\Gamma(N)$ for some N. One can generalize equation (2.52) by integrating around the boundary of a fundamental domain for Γ'.

2.5.1 Summary of some basic results on modular forms

In this section we take the multiplier system to be trivial. That is $M(\gamma) = 1$ is a 1×1 matrix.

We have included the condition "nearly holomorphic" above because in the theory of modular forms discussed in the math literature an important growth condition is placed on the function $f(\tau)$, namely, that they do not have exponential growth for $\tau \to i\infty$. That is, there are constants c, k so that $|f(\tau)| \le c(\mathrm{Im}\tau)^k$ for $\mathrm{Im}\tau \to \infty$, or, equivalently, $F(n) = 0$ for $n < 0$ in (2.51). With this added condition the function is known as a *modular form*. We can describe the space of modular forms very explicitly. Let $M_w(\Gamma)$ denote the vector space over $\mathbb{C}$ of modular forms of weight w.

The first thing to show is that some $M_w(\Gamma)$ are nonempty. We do this by constructing the Eisenstein series, defined by

$$G_w(\tau) := \sum \frac{1}{(m\tau + n)^w} \tag{2.53}$$

The sum is on integers $(m, n) \in \mathbb{Z}^2 - \{(0, 0)\}$. The sum converges absolutely for $w > 2$. Moveover, the sum vanishes for w an odd integer. Thus we restrict attention to w even and $w \ge 4$. The modularity is obvious by direct substitution.

For later purposes it is very useful to rewrite (2.53). Let Γ_∞ be the subgroup of modular transformations generated by T. We may identify $\Gamma_\infty \backslash \Gamma$ with the set of pairs of relatively prime integers (c, d). Note that $ad - bc = 1$, so (c, d) are relatively prime and since

$$\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + \ell c & b + \ell d \\ c & d \end{pmatrix} \tag{2.54}$$

we can map a coset unambiguously to the pair (c, d). Conversely, given a pair (c, d) we can always find a corresponding (a, b) so that $ad - bc = 1$ and construct an element of $SL(2, \mathbb{Z})$ Different choices of (a, b) are related by (2.54). Now, in view of this observation we can rewrite:

$$G_w(\tau) = \zeta(w) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, \tau)^{-w} \tag{2.55}$$

Using the cocycle identity (2.49) we immediately verify that G_w is a modular form of weight w, provided it converges. It is common in the literature to define $G_w(\tau) = 2\zeta(w) E_w(\tau)$ so that the Fourier expansion $E_w(\tau)$ begins with 1. With a little work one can derive the Fourier expansion

$$E_{2k}(q) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \tag{2.56}$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the sum of the kth powers of the divisors of n and B_{2k} is the Bernoulli number.

Note that modular forms form a $\mathbb{Z}$-graded ring denoted

$$M_*(\Gamma) = \oplus_{w\in\mathbb{Z}} M_w(\Gamma) \tag{2.57}$$

The first basic theorem in the theory of modular forms states that this ring is a polynomial ring generated by Eisenstein series E_4, E_6 of weights $4, 6$:

$$M_*(\Gamma) = \mathbb{C}[E_4, E_6] \tag{2.58}$$

Thus, $M_w = 0$ for $w < 4$, except for $M_0 = \mathbb{C}$, generated by the constant function.

This theorem is proven by systematically applying the key result (2.52). For details see [32] or [22].

To give a flavor of the proof note that for a modular form $v_p(f)$ must be nonnegative integers. Thus, it immediately follows that $M_w = 0$ for $w < 0$. Moreover, the sum rule can only be saturated for $w = 0$ if all $v_p(f) = 0$. Therefore f is constant, so $M_0 = \mathbb{C} \cdot 1$. For M_2 there is no way to satisfy the sum rule, so $M_2 = 0$. For $w = 4$ the only solution is $v_\rho = 1$ with all other $v_p = 0$. Thus, M_4 is one-dimensional. It must be generated by E_4, and moreover we learn that E_4 has a simple zero at $\tau = \rho$, and no other zeroes in $\mathcal{F}$. Similarly, M_6 is generated by E_6 which has a simple zero at $\tau = i$. In a similar way we find that M_8 is generated by E_4^2 and M_{10} is generated by $E_4 E_6$.

Something new happens at $w = 12$. Note that at weight $w = 12$ since E_4 and E_6 are equal to 1 for $q = 0$, if we define Δ by

$$E_4^3 - E_6^2 = (12)^3 \Delta \tag{2.59}$$

then Δ is manifestly holomorphic, and clearly has a zero at $q = 0$. Therefore, by (2.52) it has a first order zero at $q = 0$ and no other zeroes in the upper half-plane.

Now, if f is any modular form of weight $w = 12$ then

$$\frac{f - \hat{f}(0) E_{12}}{\Delta} \tag{2.60}$$

is in M_0, and hence a constant, so

$$M_{12} = \langle \Delta, E_{12} \rangle = \langle \Delta, E_4^3 \rangle = \langle \Delta, E_6^2 \rangle \tag{2.61}$$

The same argument in fact shows that

$$M_w = \Delta \cdot M_{w-12} \oplus \langle E_w \rangle \tag{2.62}$$

and hence it follows that

$$\dim M_w = \begin{cases} [\frac{w}{12}] & w = 2 \mathrm{mod} 12 \\ [\frac{w}{12}] + 1 & w \neq 2 \mathrm{mod} 12 \end{cases} \tag{2.63}$$

Remarks

1. A modular form with $F(0) = 0$, i.e. a form which vanishes for $q \to 0$ is called a "cusp form." Define $S_w \subset M_w$ the space of cusp forms. Then $M_w = S_w \oplus \mathbb{C}E_w$.
2. Given the transformation properties of η we see that

$$\Delta = \eta^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \tag{2.64}$$

manifestly showing that it is nonvanishing on $\mathbb{H}$.
3. Similarly, we can construct the j-function

$$J(\tau) = \frac{E_4^3}{\Delta} = \frac{E_6^2}{\Delta} + (12)^3 = q^{-1} + 744 + 196884q + \cdots \tag{2.65}$$

2.5.2 Negative weight and the polar part

The examples of partition functions above show that we might wish to relax the standard growth condition on modular forms since we consider inverse powers of η functions. These have a singularity at $q = 0$. Moreover, we should consider modular forms of negative weight.

Now, a simple, but crucial observation for physical applications is that *for negative weight nearly holomorphic modular forms the polar part of the form uniquely determines the entire form.*

In physical terms this will mean that for holomorphic partition functions *the degeneracies of polar states completely determine the entire spectrum.*

One way to see this is to use the identity (2.52). We see that negative weight forces some terms $v_p(f)$ to be negative. By definition nearly holomorphic functions have $v_p(f) \geq 0$ for $p \neq \infty$, so there must be a polar piece. Now, if we have two forms $f(\tau)$ and $\tilde{f}(\tau)$ with the same polar part then $f(\tau) - \tilde{f}(\tau)$ has no polar part, and therefore must vanish, so $f = \tilde{f}$. Note that this conclusion is quite false if we drop holomorphy, or modularity, or even if we insist on holomorphy and modularity but consider the case $w > 0$. In this last case, we can always modify the nonpolar degeneracies by adding a cuspform.

This argument can be generalized to vector-valued modular forms where $M(\gamma)$ becomes diagonal on a congruence subgroup. A word of warning: The crucial identity (2.52) becomes more complicated because the fundamental domains are more complicated. For a nifty online program that draws fundamental domains of congruence subgroups see Helena Verrill's program at http://www.math.lsu.edu/ verrill/fundomain/ .

3. Extended supersymmetry and the Elliptic Genus

3.1 $\mathcal{N} = 2$ superconformal symmetry and spectral flow

3.1.1 $\mathcal{N} = 2$ superconformal algebra

Let us now consider a conformal field theory with $\mathcal{N} = 2$ supersymmetry. That is, the Hilbert space $\mathcal{H}$ is a representation of the $\mathcal{N} = 2$ superconformal algebra. The holomorphic currents which generate the algebra are the energy momentum tensor $T(z)$, the two supercurrents $G^{\pm}(z)$ and a dimension one $U(1)$ current $J(z)$. Using standard definitions of the modings the Lie algebra relations are the Virasoro relations for L_n, in addition the other currents are Virasoro primaries

$$[L_n, J_{n'}] = -n' J_{n+n'} \tag{3.1}$$

$$[L_n, G_r^{\pm}] = \left(\frac{n}{2} - r\right) G_{n+r}^{\pm} \tag{3.2}$$

The $U(1)$ current algebra satisfies

$$[J_n, J_{n'}] = \frac{c}{3} n \delta_{n+n',0}$$

and the supercurrents $G^{\pm}$ have charges ± 1:

$$[J_n, G_r^{\pm}] = \pm G_{n+r}^{\pm}$$

Finally, we have the supersymmetry algebra:

$$[G_r^{\pm}, G_s^{\pm}]_+ = 0$$

because there are no elements of charge ± 2 and, most importantly,

$$[G_r^{\pm}, G_s^{\mp}]_+ = 2L_{r+s} \pm (r-s) J_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r+s,0} \tag{3.3}$$

Here $n, m \in \mathbb{Z}$ but the mode numbers r, s of G_r^+, G_s^- can be in a nontrivial $\mathbb{Z}$-torsor $r \in \mathbb{Z} + a$, $s \in \mathbb{Z} - a$ for any real number a. The algebra for $a = \frac{1}{2} \mathrm{mod} \mathbb{Z}$ is known as the Neveu-Schwarz (NS) algebra, while that for $a = 0 \mathrm{mod} \mathbb{Z}$ is known as the Ramond (R) algebra.

It will be convenient to define $\hat{c}$ and m by:

$$c = 3\hat{c} = 6m \tag{3.4}$$

Supersymmetric sigma models with a Kähler target space have $\mathcal{N} = (2,2)$ supersymmetry. If the target space X has complex dimension d then $\hat{c} = d$. Supersymmetric sigma models with a hyperkähler target space have d even, and hence m integral. These models in fact have extended $\mathcal{N} = (4,4)$ supersymmetry. There are also very interesting theories with $(0,2)$ and $(0,4)$ supersymmetry.

For simplicity, in what follows we will assume that m is integral, and that the spectrum of J_0 is integral.

3.1.2 Spectral flow isomorphism

The $\mathcal{N} = 2$ algebra makes sense for $r \in \mathbb{Z} + a, s \in \mathbb{Z} - a$. In fact, the algebras are isomorphic for all values of a thanks to the so-called *spectral flow isomorphism* which maps [31]

$$G_{n \pm a}^{\pm} \to G_{n \pm (a+\theta)}^{\pm} \tag{3.5}$$

$$L_n \to L_n + \theta J_n + \frac{c}{6}\theta^2 \delta_{n,0} \tag{3.6}$$

$$J_n \to J_n + \frac{c}{3}\theta \delta_{n,0}. \tag{3.7}$$

where θ is any real number.

Note that the combination

$$4mL_0 - J_0^2 \tag{3.8}$$

is spectral flow invariant.

Spectral flow allows us to relate traces in the NS and R sectors. Define

$$Z_{RR}(\tau, z; \bar{\tau}, \bar{z}) := \mathrm{Tr}_{\mathcal{H}_{RR}} q^{L_0 - c/24} e^{2\pi i z J_0} \bar{q}^{\tilde{L}_0 - c/24} e^{2\pi i \bar{z} \tilde{J}_0} e^{i\pi(J_0 - \tilde{J}_0)} \tag{3.9}$$

The factor of $e^{i\pi(J_0-\tilde{J}_0)}$ is inserted for later convenience. Z_{NSNS} is defined similarly. The spectral flow image of a partition function is defined by substituting the above transformations:

$$(SF_\theta \widetilde{SF}_{\tilde{\theta}} Z) := e\left(m\theta^2\tau + 2m\theta z\right) e\left(m\tilde{\theta}^2\bar{\tau} + 2m\tilde{\theta}\bar{z}\right) Z(\tau, z + \theta\tau; \bar{\tau}, \bar{z} + \tilde{\theta}\bar{\tau}) \tag{3.10}$$

Therefore, spectral-flow invariant theories must satisfy

$$Z_{RR} = (SF_\theta \widetilde{SF}_{\tilde{\theta}}) Z_{RR} \qquad \theta, \tilde{\theta} \in \mathbb{Z} \tag{3.11}$$

$$Z_{NSNS} = (SF_\theta \widetilde{SF}_{\tilde{\theta}}) Z_{RR} \qquad \theta, \tilde{\theta} \in \mathbb{Z} + \frac{1}{2} \tag{3.12}$$

$$\tag{3.13}$$

3.1.3 Highest weight states, primary, and chiral primary states

We will need some results on the representation theory of the $\mathcal{N} = 2$ superconformal algebra, and in particular the constraints of unitarity. These were worked out fully by Boucher, Friedan and Kent in [6].

In the NS sector an $\mathcal{N} = 2$ primary field satisfies:

$$G_r^\pm |h, q\rangle = 0 \qquad r > 0 \tag{3.14}$$

$$L_n |h, q\rangle = 0 \qquad n > 0 \tag{3.15}$$

$$J_n |h, q\rangle = 0 \qquad n > 0 \tag{3.16}$$

$$L_0 |h, q\rangle = h|h, q\rangle \tag{3.17}$$

$$J_0 |h, q\rangle = q|h, q\rangle \tag{3.18}$$

Such a state generates a highest weight representation $V_{h,q}$. By spectral flow we obtain corresponding highest weight representations in the Ramond sector.

Unitarity implies

$$\begin{aligned} 0 &\leq \| G^\pm_{-1/2} |h, q\rangle \|^2 \\ &= \langle h, q| G^\mp_{1/2} G^\pm_{-1/2} |h, q\rangle \\ &= \langle h, q| 2L_0 \mp J_0 |h, q\rangle \\ &= 2h \mp q \end{aligned} \tag{3.19}$$

and hence we get our first example of a BPS bound:

$$h \geq \frac{|q|}{2} \tag{3.20}$$

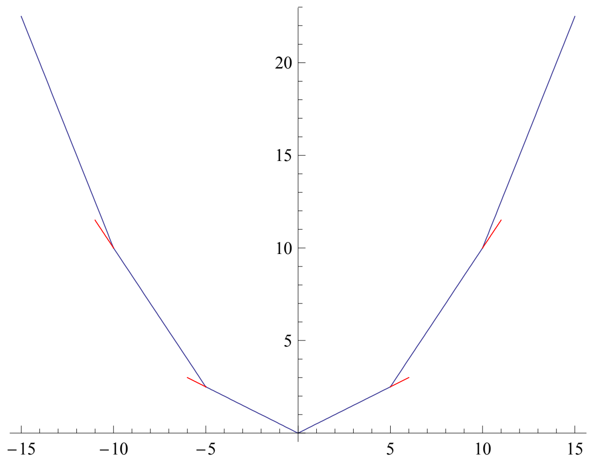

Figure 4: Unitarity region for NS sector highest weight representations of the $\mathcal{N}=2$ algebra. The chiral primaries lie on the line $h=q/2$, $0\le q\le m$. The antichiral primaries on the line $h=-q/2$, $-m\le q\le 0$. We have illustrated the case $m=3$."

Highest weight (BPS) states which are annihilated by $G^+_{-1/2}$ saturate the bound $h=\frac{|q|}{2}$. They are also known as *chiral primaries*. Similarly, BPS states annihilated by $G^-_{-1/2}$ are known as *anti-chiral primaries*. The only BPS state which is both chiral and antichiral is the vacuum $h=q=0$.

A similar computation with $0\le\parallel G^\pm_{-3/2}|h,q\rangle\parallel^2$ leads to the bound

$$2h\mp 3q+4m\ge 0 \tag{3.21}$$

For a chiral primary field this implies $q\le 2m$. The bound is saturated by a unique chiral primary with $G^+_{-3/2}|h=m,q=2m\rangle=0$. This is the spectral flow of the vacuum by one unit.

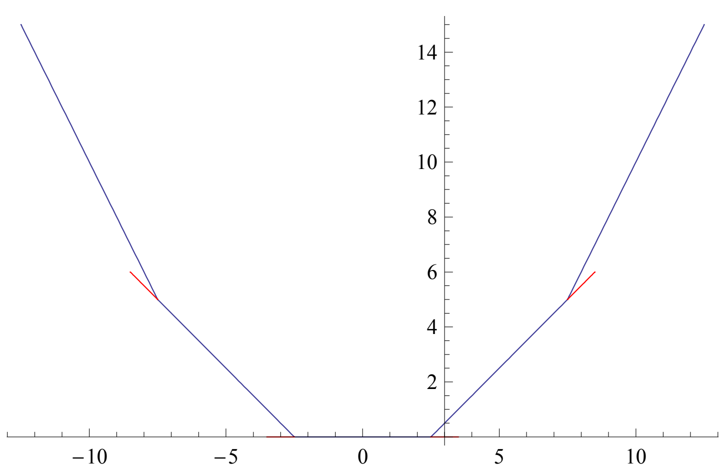

Figure 5: Unitarity region for the Ramond sector. Ramond groundstates are at $h=m/4$, $-m\le q\le m$. We have illustrated $m=3$.

Under spectral flow by $\theta = +\frac{1}{2}$ we have:[6]

$$G^+_{-1/2}|h,q\rangle_{NS} = 0 \quad \overset{\theta=+1/2}{\rightarrow} \quad G^+_0|h - \frac{q}{2} + \frac{m}{4}, q-m\rangle_R = 0 \tag{3.22}$$

$$G^-_{1/2}|h,q\rangle_{NS} = 0 \quad \overset{\theta=+1/2}{\rightarrow} \quad G^-_0|h - \frac{q}{2} + \frac{m}{4}, q-m\rangle_R = 0 \tag{3.23}$$

giving us the Ramond sector groundstates.

In the Ramond sector positivity of $\parallel G^\pm_0|h,q\rangle \parallel^2$ bounds L_0 below by $\frac{c}{24}$. Under spectral flow by $\theta = +1/2$ to the R sector the BPS states in the NS sector map to R groundstates with $L_0 = c/24$ and $-m \le q \le m$, as shown in fig. 5.

Remarks

1. The full analysis of unitarity is quite intricate. We will just state the conjecture of [6]. We assume $c \ge 3$, i.e. $m \ge \frac{1}{2}$. Then, for the NS sector the unitary representations are of "type A3" or "type A2." The type A3 representations lie within a discrete approximation to a parabola. This parabola is given by

$$(4m-2)h = q^2 \tag{3.24}$$

On this parabola we draw chords joining the successive points $(q = (2m-1)s, h = \frac{1}{2}(2m-1)s^2)$, for $s \in \mathbb{Z}$. Then (q,h) must lie in the closed convex region defined by the chords. In addition, there can be representations of type A2 which lie on the line segments

$$2h - (2s-1)q + (2m-1)(s^2 - s) = 0 \qquad (2m-1)s \le q \le (2m-1)s + 1 \tag{3.25}$$

for $0 \le s$, and their charge conjugate images. A similar picture holds in the Ramond sector. The discrete approximation to the parabola $(4m-2)(h-m/4) - q^2 + (m-1/2)^2 = 0$ is drawn by drawing chords between successive points $(q = (2m-1)(s+1/2), h - m/4 = \frac{1}{2}s(s+1)(2m-1))$. The closed convex region contains the unitary representations of type P3. In addition there can be representations of type P2 on the segments

$$2(h-m/4) - 2sq + (2m-1)s^2 = 0 \qquad (2m-1)(s+1/2) \le q \le (2m-1)(s+1/2) + 1 \tag{3.26}$$

for $s \ge 0$, and their charge conjugate images.

2. A proof of the determinant formulae of [6] was given in [20]. As far as we know, full proofs of the unitarity constraints have never been published.

[6]Warning: In general, spectral flow does not take highest weight vectors to highest weight vectors. This is obvious from the change in the moding of $G^\pm_r$.

3.1.4 Path integral interpretation and modular invariance

As before, there is a path integral interpretation of these partition functions. In the interest of brevity we will be sketchy here. Schematically it takes the form:

$$Z_\epsilon = \left\langle e^{2\pi i \int_{E_\tau} \left(A^{0,1} J + A^{1,0} \tilde{J}\right)} \right\rangle_\epsilon \tag{3.27}$$

Here $E_\tau := \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ is an elliptic curve and

$$\begin{aligned} A &= A^{0,1} d\bar{\xi} + A^{1,0} d\xi \\ &= \frac{i}{2\mathrm{Im}\tau}\left(\bar{z} d\xi - z d\bar{\xi}\right) \end{aligned} \tag{3.28}$$

is a flat 1-form. We introduce complex coordinate $\xi = \sigma^1 + \tau\sigma^2$ on E_τ as in section ***. The subscript ϵ refers to the the (left and right) spin structures.

Now, the currents $J(\xi)d\xi$ and $\tilde{J}(\xi)d\bar{\xi}$ have singularities in their operator product expansions:

$$J(\xi_1)J(\xi_2) \sim \frac{2m}{(\xi_1-\xi_2)^2} + \cdots \tag{3.29}$$

and therefore (3.27) requires regularization and renormalization.

♣ Explain this better by subtracting the square of the prime form. ♣

Now let us consider diffeomorphism invariance. In general, diffeomorphisms act non-trivially on the set of spin structures. There are four spin structures on the torus, only one of which is nonbounding. This must be prserved by diffeomorphisms. It is the one corresponding to RR boundary conditions for the fermions both on the left and the right. Let us take this pair of spin structures.

Next, we have $f^*(d\xi) = (c\tau+d)d\xi$ so z must transform under modular transformations as:

$$z \to \frac{z}{c\tau+d} \tag{3.30}$$

Now, if the underlying theory is diffeomorphism invariant then there is a regularization which makes Z_ϵ diffeomorphism invariant. It turns out that this regularization involves a contact term between J and $\tilde{J}$, leading to an overall factor $\sim \exp const. \int A^{1,0}\wedge A^{0,1}$. The net result is that the diffeomorphism invariant partition function is

$$e^{-\pi m \frac{(z-\bar{z})^2}{\mathrm{Im}\tau}} Z_{RR}(\tau, z; \bar{\tau}, \bar{z}) \tag{3.31}$$

and the modular transformation law of the partition function is:

$$Z_{RR}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{a\bar{\tau}+b}{c\bar{\tau}+d}, \frac{\bar{z}}{c\bar{\tau}+d}\right) = e\left(m\frac{cz^2}{c\tau+d}\right)\overline{e\left(m\frac{cz^2}{c\tau+d}\right)} Z_{RR}(\tau, z; \bar{\tau}, \bar{z}) \tag{3.32}$$

(To see how this works in detail in a representative example work out the partition function of a chiral fermion coupled to $\bar{A}$. For details see, for example, [3].)

3.2 The $(2,2)$ elliptic genus

Of particular importance is the *Witten index*, this is a specialization of the above partition functions which counts the BPS representations of the superconformal algebra. In the present context it becomes the elliptic genus.

The elliptic genus for a $(2,2)$ CFT $\mathcal{C}$ is defined to be:

$$\chi(\tau, z; \mathcal{C}) := Z_{RR}(\tau, z; \bar{\tau}, 0) \tag{3.33}$$

$$= \mathrm{Tr}_{RR} e^{2\pi i\tau(L_0 - c/24)} e^{2\pi i z J_0} e^{-2\pi i\bar{\tau}(\tilde{L}_0 - c/24)} (-1)^F \tag{3.34}$$

where $(-1)^F = \exp[i\pi(J_0 - \tilde{J}_0)]$ is ± 1, given our assumption of integral m and $U(1)$ spectrum.

The key to understanding the elliptic genus is that in a Ramond sector highest weight representation $V_{h,q}$ we have

$$\mathrm{Tr}_{V_{h,q}} q^{L_0 - c/24} e^{i\pi J_0} = \begin{cases} e^{i\pi q} & h = \frac{c}{24} = \frac{m}{4} \\ 0 & h > \frac{c}{24} = \frac{m}{4} \end{cases} \tag{3.35}$$

The elliptic genus satisfies the following important properties:

- First, thanks to (3.35) $\chi(\tau, z)$ is not a function of $\bar{\tau}$. Moreover, in unitary theories with discrete spectrum, it will be holomorphic for $\tau \in \mathcal{H}$ and entire in z.

- Next, the modular transformation properties of the path integral (3.32) leads to the the transformation laws for $\gamma \in SL(2, \mathbb{Z})$:

$$\chi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = e^{2\pi i m \frac{cz^2}{c\tau + d}} \chi(\tau, z) \tag{3.36}$$

- Finally, the phenomenon of spectral flow is encoded in:

$$\chi(\tau, z + \theta\tau + \theta') = e^{-2\pi i m(\theta^2\tau + 2\theta z)} \chi(\tau, z) \qquad \theta, \theta' \in \mathbb{Z} \tag{3.37}$$

Remarks

1. The elliptic genus can be introduced for any theory with supersymmetry. It was introduced for $(0,1)$ theories in [REFS]. The systematic investigation of the properties for $\mathcal{N} = 2$ theories was begun in [21].

2. It is important that we are assuming integral m and $U(1)$ charges, otherwise there are some modifications on these conditions. See [21].

3.3 Jacobi forms

Definition A *Jacobi form* $\phi(\tau, z)$ of weight w and index m is a function which is holomorphic in $z \in \mathbb{C}$, and in $\tau \in \mathcal{H}$ and satisfies the identities:

$$\phi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^w e^{2\pi i m \frac{cz^2}{c\tau+d}}\phi(\tau,z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{3.38}$$

$$\phi(\tau, z+\theta\tau+\theta') = e^{-2\pi i m(\theta^2\tau+2\theta z)}\phi(\tau,z) \qquad \theta,\theta' \in \mathbb{Z} \tag{3.39}$$

The standard reference is the book by Eichler and Zagier [12]. In this book only integral values of m are considered.

From equations (3.38) and (3.39) it follows that $\phi(\tau, z)$ has a Fourier expansion in both variables, so we can define Fourier coefficients:

$$\phi(\tau,z) = \sum_{n\in\mathbb{Z},\ell\in\mathbb{Z}} c(n,\ell)q^n y^\ell \tag{3.40}$$

In the math literature [12] again one specifies growth conditions at infinity. Strictly speaking a Jacobi form is reserved for functions such that $c(n,\ell) = 0$ unless $4mn - \ell^2 \geq 0$. As with modular forms, this is too restrictive for physical applications. The mathematical notion which fits perfectly with applications to unitary $\mathcal{N} = 2$ theories is that of a *weak Jacobi form*. This is a holomorphic function transforming as in (3.38) and (3.39) which in addition satisfies $c(n,\ell) = 0$ unless $n \geq 0$.

There are two main structure theorems for Jacobi forms that we will need: The theta function decomposition and the structure of the ring of Jacobi forms.

Thanks to (3.39) the coefficients $c(n,\ell)$ satisfy

$$c(n,\ell) = c(n+\ell s+ms^2, \ell+2ms) \tag{3.41}$$

where s is any integer. It follows from (3.41) that a weak Jacobi form has $c(n,\ell) = 0$ if $4nm - \ell^2 < -m^2$. We will return to this point below.

An important consequence of (3.41) is that the $c(n,\ell)$ are in fact only a function of the combination

$$p = 4mn - \ell^2 \tag{3.42}$$

and the discrete variable $\nu := \ell \text{mod} 2m$, so we can write:

$$c(n,\ell) = c_\nu(p) \tag{3.43}$$

To prove this let us fix a fundamental domain for translation by $2m$ and write $\ell = \nu + 2ms_0$, with integral s_0, and $-m+1 \leq \nu \leq m$. Then we can put $s = -s_0$ in (3.41) and write:

$$c(n,\ell) = c(n-\nu s_0 - ms_0^2, \nu) = c(\frac{4mn-\ell^2+\nu^2}{4m}, \nu) \tag{3.44}$$

Thanks to (3.41) we can write the z dependence of the elliptic genus exactly in terms of theta functions. We sum over lattice points (n,ℓ) on the parabola $4nm - \ell^2 = const$ and then sum over the constants. If we fix μ and sum over those points with $\ell = \mu \mathrm{mod} 2m$, the sum over the parabola leads to a theta function of level m, denoted $\Theta_{\mu,m}(z,\tau)$ and defined by:

$$\Theta_{\mu,m}(z,\tau) := \sum_{\ell\in\mathbb{Z},\ell=\mu \mathrm{mod} 2m} q^{\ell^2/(4m)} y^\ell = \sum_{n\in\mathbb{Z}} q^{m(n+\mu/(2m))^2} y^{(\mu+2mn)} \tag{3.45}$$

We thus arrive at the theta function decomposition

$$\phi(\tau,z) = \sum_{\mu \mathrm{mod} 2m} h_\mu(\tau)\Theta_{\mu,m}(z,\tau) \tag{3.46}$$

where

$$h_\mu(\tau) = \sum_{p=-\mu^2 \mathrm{mod} 4m} c_\mu(p) q^{\frac{p}{4m}} \tag{3.47}$$

We conclude that to give an elliptic genus of weight w is equivalent to giving a vector-valued modular form of weight $w-1/2$ transforming contragrediently to the level m theta functions. The latter transform as in

$$\Theta_{\mu,m}(z,\tau+1) = e^{2\pi i \frac{\mu^2}{4m}}\Theta_{\mu,m}(z,\tau) \tag{3.48}$$

$$\Theta_{\mu,m}(-z/\tau,-1/\tau) = (-i\tau)^{1/2} e^{2\pi i m z^2/\tau} \sum_{\nu \mathrm{mod} 2m} \frac{1}{\sqrt{2m}} e^{2\pi i \frac{\mu\nu}{2m}} \Theta_{\nu,m}(z,\tau) \tag{3.49}$$

♣ Relation to Heisenberg group. General tmn law. Give contragredient rep? ♣

As with modular forms, the weak Jacobi forms $\tilde{J}_{w,m}$ form a bigraded ring: $\tilde{J}_{*,*} = \oplus_{w,m}\tilde{J}_{w,m}$. The main fact is

Theorem. $\tilde{J}_{*,*}$ is a polynomial ring over $M_*(\Gamma)$ on two generators $\tilde{\phi}_{-2,1} \in \tilde{J}_{-2,1}$ and $\tilde{\phi}_{0,1} \in \tilde{J}_{0,1}$:

$$\tilde{J}_{*,*} = \mathbb{C}[E_4, E_6, \tilde{\phi}_{-2,1}, \tilde{\phi}_{0,1}] \tag{3.50}$$

This is proved in [12] along the following lines. The above generators allow us to describe an explicit map

$$M_w(\Gamma) \oplus M_{w+2}(\Gamma) \oplus \cdots \oplus M_{w+2m}(\Gamma) \to \tilde{J}_{w,m} \tag{3.51}$$

One shows that $\tilde{\phi}_{-2,1}, \tilde{\phi}_{0,1}$ are linearly independent, so the map is injective.

Conversely, given a Jacobi form $\phi(\tau,z)$ one can write its Taylor series about $z=0$:

$$\phi(\tau,z) = \sum_{\nu=0}^{\infty} \chi_\nu(\tau) z^\nu \tag{3.52}$$

Then, from (3.38) we see that $\chi_\nu(\tau)$ transform as modular forms of weight $w+\nu$ plus lower order terms. This leads to a triangular system of equations from which one can extract true modular forms ξ_ν, and one uses these to define a map

$$\tilde{J}_{w,m} \to M_w(\Gamma) \oplus M_{w+2}(\Gamma) \oplus \cdots \oplus M_{w+2m}(\Gamma) \tag{3.53}$$

which is also injective. Thus, the spaces must be isomorphic. ♠

3.4 The singleton decomposition

The theta function decomposition (3.46) has a nice physical interpretation. First of all, we use the $U(1)$ current to introduce chiral bosons

$$J = i\sqrt{2m}\partial\phi(\xi) \qquad \tilde{J} = -i\sqrt{2m}\bar{\partial}\tilde{\phi}(\bar{\xi}) \tag{3.54}$$

The theory "factorizes" into a theory of $U(1)$-neutral operators and the theory of this free boson.

In the context of the AdS/CFT correspondence the chiral bosons $\phi, \tilde{\phi}$ correspond to "singleton degrees of freedom," in the bulk supergravity. The currents $J, \tilde{J}$ are dual to $U(1)$ gauge fields in the bulk with Chern-Simons terms. The corresponding gauge modes are "topological" in the bulk, but have physical propagating degrees of freedom on the boundary. These are known as "singleton degrees of freedom." They are similar to the edge states in the fractional quantum Hall effect.

The partition functions for a chiral boson of radius $R^2 = m$ are given by (2.39) where μ gives the $U(1)$ charge modulo $2m$. Thus the partition function should be written as

$$\sum_{\mu} \tilde{h}_{\mu} \frac{\Theta_{\mu,m}(z,\tau)}{\eta} \tag{3.55}$$

where $\tilde{h}_{\mu}$ are the partition functions of the neutral "bulk" degrees of freedom. In this way we recover the decomposition (3.46).

Remarks

1. The singleton decompositions of partition functions in AdS/CFT is discussed in more detail in [39, 24, 19, 27, 4].

3.5 Examples from supersymmetric sigma models

In general, a supersymmetric sigma model with a Kähler target space X has $(2,2)$ supersymmetry.

Under modular transformations one makes a chiral transformation on the worldsheet fermions. There is an gravitational anomaly in the nonlinear sigma model unless one restricts to $c_1(X) = 0$, i.e. to Calabi-Yau manifolds. [REFS], so we will restrict to this case. In this case Fourier coefficients of the elliptic genus can be computed explicitly in terms of the by the Chern-numbers of the holomorphic tangent bundle of X by [21][17]

$$\chi(\tau,z)_X = \int_X \prod_{j=1}^{2m} \frac{\vartheta_1(\tau, z + \frac{\xi_j}{2\pi i})}{\vartheta_1(\tau, \frac{\xi_j}{2\pi i})} \xi_j, \tag{3.56}$$

where the ξ_j are defined by

$$c(T_X) = 1 + c_1(T_X) + \ldots c_{2m}(T_X) = \prod_{j=1}^{2m} (1 + \xi_j). \tag{3.57}$$

In general, the $q \to 0$ limit is given in terms of the Hodge numbers of the target space X:

$$\chi(\tau, z) \to \sum_{i,j=0}^{2m} (-1)^{i+j} h^{i,j}(X) y^{j-m} := \sum_{j=0}^{2m} \chi_j y^{j-m} \tag{3.58}$$

where χ_j are known as *Hirzebruch genera.*

As a nice consistency check take the limit $q \to 0$ in (3.56) using $\vartheta_1(z|\tau) \to -2q^{1/8}\sin(\pi z)$ to get

$$\int_X \prod_{j=1}^{2m} \left(\frac{e^{i\pi z + \frac{1}{2}\xi_j} - e^{-i\pi z - \frac{1}{2}\xi_j}}{e^{\frac{1}{2}\xi_j} - e^{-\frac{1}{2}\xi_j}} \right) \prod \xi_j = y^{-m} \int_X \mathrm{ch}(\Lambda_y T^* X) \prod \frac{\xi_j}{1 - e^{-\xi_j}} \tag{3.59}$$

where $\Lambda_y T^* X = \sum_j (-y)^j \Lambda^j T^* X$, which agrees with (3.58) by the index theorem.

The simplest Calabi-Yau manifold is the elliptic curve, but for this case $\chi = 0$. At complex dimension 2, corresponding to $m = 1$ there is the abelian surface T^4 and the $K3$ surface. Again $\chi = 0$ for T^4, because of fermion zeromodes, so the first interesting nontrivial case is the K3 elliptic genus. Because of the topological nature of the genus it can be computed in an orbifold limit of a Kummer surface, where we identify $K3 = T^4/\mathbb{Z}_2$. The computation for the orbifold is straightforward and leads to the result:

$$\chi(\tau, z; K3) = 8\left(\Big(\frac{\vartheta_2(z|\tau)}{\vartheta_2(0|\tau)}\Big)^2 + \Big(\frac{\vartheta_3(z|\tau)}{\vartheta_3(0|\tau)}\Big)^2 + \Big(\frac{\vartheta_4(z|\tau)}{\vartheta_4(0|\tau)}\Big)^2 \right) \tag{3.60}$$

By the general structure theorem on weak Jacobi forms it is clear that this must be proportional to the generator $\tilde{\phi}_{0,1}$, and by comparing the Fourier expansion

$$\tilde{\phi}_{0,1} = (y + 10 + y^{-1}) + q(10y^2 - 64y + 108 - 64y^{-1} + 10y^{-2}) + \mathcal{O}(q^2) \tag{3.61}$$

we see that $\chi(\tau, z; K3) = 2\tilde{\phi}_{0,1}$.

Remarks

1. If the superconformal field theory arises from a sigma model with target space X the elliptic genus has the interpretation of being the character-valued index of the Dirac-Ramond operator on the loopspace LX. ♣ MORE DETAILS

3.6 Symmetric products and the product formula

An important construction in the mathematics of supersymmetric black holes is the symmetric product construction. If $\mathcal{C}$ is a superconformal field theory then $\mathcal{C}^{\otimes N}$ can be given a superconformal structure using the diagonal combination of $T, G^{\pm}, J$, and taking the graded tensor product. Clearly the symmetric group acts on this conformal field theory, commuting with the superconformal algebra, so the orbifold

$$\mathrm{Sym}^N \mathcal{C} := \mathcal{C}^{\otimes N}/S_N \tag{3.62}$$

is an $\mathcal{N} = 2$ theory.

Now, the twisted sector associated with a cycle of length n is easily visualized as the string on a circle which is the connected n-fold covering of $S^1 \to S^1$. That is, it is a "long string" of length $2\pi n$. We denote the Hilbert space of this long string as $\mathcal{H}^{(n)}(\mathcal{C})$.

Accordingly, the spectrum of energies,i.e. on this string is rescaled to by $1/n$. The RR sector of the symmetric product $\mathcal{N}=2$ theory is:

$$\mathcal{H}_{RR}(\mathrm{Sym}^N(\mathcal{C})) = \oplus_{(n)^{\ell_n}} \otimes_n \mathrm{Sym}^{\ell_n}(\mathcal{H}^{(n)}_{RR}(\mathcal{C})) \tag{3.63}$$

where we sum over cycle decompositions $(n)^{\ell_n}$ of elements of the symmetric group, that is, we sum over partitions $\sum n\ell_n = N$.

We will now give two interesting formulae from which one can extract the spectrum of these symmetric product theories. [7]

It turns out to be very useful to consider the generating function of all symmetric product partition functions because we can write:

$$\mathcal{Z} := 1 + \sum_{N\geq 1} p^N \mathrm{Tr}_{\mathcal{H}(\mathrm{Sym}^N(\mathcal{C}_0))} q^H y^J \bar{q}^{\bar{H}} \bar{y}^{\bar{J}} = \prod_{n=1}^{\infty} \sum_{\ell_n=0}^{\infty} p^{n\ell_n} \mathrm{Tr}_{\mathrm{Sym}^{\ell_n}(\mathcal{H}^{(n)})} q^H y^J \bar{q}^{\bar{H}} \bar{y}^{\bar{J}} \tag{3.64}$$

Here $H = L_0 - c/24, J = J_0$ etc.

Suppose we have an expansion:

$$\mathrm{Tr}_{\mathcal{H}} q^H y^J \bar{q}^{\bar{H}} \bar{y}^{\bar{J}} = \sum_{h,\ell,\tilde{h},\tilde{\ell}} c(h,\ell;\tilde{h},\tilde{\ell}) q^h y^\ell \bar{q}^{\tilde{h}} \bar{y}^{\tilde{\ell}} \tag{3.65}$$

Then we claim

$$\mathcal{Z} = \prod_{n=1}^{\infty} \prod_{h,\ell,\tilde{h},\tilde{\ell}}^{(n)} (1 - p^n q^{h/n} y^\ell \bar{q}^{\tilde{h}/n} \bar{y}^{\tilde{\ell}})^{-c(h,\ell;\tilde{h},\tilde{\ell})} \tag{3.66}$$

We can prove this as follows: The standard formula for traces in symmetric products of vector spaces gives

$$\sum_{\ell_n=0}^{\infty} p^{n\ell_n} \mathrm{Tr}_{\mathrm{Sym}^{\ell_n}(\mathcal{H}^{(n)})} q^H y^J \bar{q}^{\bar{H}} \bar{y}^{\bar{J}} = \prod_{basis \mathcal{H}^{(n)}} \frac{1}{1 - p^n q^H y^J \bar{q}^{\bar{H}} \bar{y}^{\bar{J}}} \tag{3.67}$$

where we take a product over an eigenbasis in $\mathcal{H}^{(n)}(\mathcal{C})$.

Now, the trace in the long string Hilbert space is related to the original one by

$$\mathrm{Tr}_{\mathcal{H}^{(n)}(\mathcal{C})}(q^H y^J \bar{q}^{\bar{H}} \bar{y}^{\bar{J}}) = \frac{1}{n} \sum_{b=0}^{n-1} \mathrm{Tr}_{\mathcal{H}(\mathcal{C})} \omega^b q^{\frac{1}{n}H} y^J \bar{q}^{\frac{1}{n}\bar{H}} \bar{y}^{\bar{J}} \tag{3.68}$$

where $\omega = e^{2\pi i(L_0 - \bar{L}_0)/n}$. Thus the sum on b projects to states that satisfy $h - \tilde{h} = 0 \mathrm{mod} n$. The energies are scaled by $1/n$ because the length of the string is scaled by n. From this the symmetric product formula follows.

[7] Symmetric product orbifolds were studied in [REFS??]. An important case of the symmetric product formula first appeared in [36] and the general case was given in [10].

If we have a $\mathbb{Z}_2$-graded Hilbert space then we should take a supertrace, and use the rule:

$$\sum_{n=0}^{\infty} p^n \mathrm{STr}_{\mathrm{Sym}^n(\mathcal{H})}(\mathcal{O}) = \prod_{eigenbasis\mathcal{H}_0} \frac{1}{(1-p\mathcal{O}_i)} \prod_{eigenbasis\mathcal{H}_1} (1-p\mathcal{O}_i) = \exp\left[\sum_s \frac{p^s}{s}\mathrm{STr}_{\mathcal{H}}(\mathcal{O})\right] \tag{3.69}$$

thus proving (3.66).

Applied to the elliptic genus we learn that

$$\sum_{N=0}^{\infty} p^N \chi(\tau, z; \mathrm{Sym}^N X) = \prod_{m=1}^{\infty} \prod_{n,\ell} (1-p^m q^n y^\ell)^{-c(mn,\ell)} \tag{3.70}$$

There is another nice formula in terms of *Hecke operators*. We take the logarithm of (3.64):

$$\log \mathcal{Z} = \sum_n \sum_{basis\mathcal{H}^{(n)}} \frac{1}{s} p^{ns} (q^H y^J \bar{q}^{\bar{H}} \bar{y}^{\bar{J}})^s \tag{3.71}$$

Using again (3.68) this can be written as

$$\log \mathcal{Z} = \sum_{N=1}^{\infty} p^N T_N Z \tag{3.72}$$

where

$$T_N Z := \frac{1}{N} \sum_{N=ns} \sum_{b=0}^{n-1} Z(\frac{s\tau+b}{n}, y^s; \frac{s\bar{\tau}+b}{n}, \bar{y}^s) \tag{3.73}$$

Using (3.69) we see that this also holds for the case of the supertrace.

3.7 Some Remarks Elliptic genera for other superconformal algebras

♣ Need to improve this section ♣

There are interesting extensions of the $\mathcal{N}=2$ superconformal algebras.

1. $\mathcal{N}=4$ algebra. In this case, in addition to the stress energy, we have and $SU(2)$ current algebra $J^i(z)$, $i=1,2,3$, at level k. There are four supercharges $G^a(z)$, $a=1,\dots 4$. We can think of a as an $so(4)=su(2)\oplus su(2)$ index and identify one $su(2)$ summand.

The representation theory is more constrained. $c=6k$, with k and integers. One can still define the elliptic genus identifying J_0 with $2J_0^3$ (with integral spectrum).

2. The $\mathcal{N}=4$ algebra $\mathcal{A}_\gamma$ has two $SU(2)$ current algebras $J^{i,\pm}(z)$ and hence two levels $k^\pm$. In addition to the four supercharges $G^a(z)$ transforming in the $(2,2)$ of $SU(2)\times SU(2)$ there is a multiplet of fermionic operators Q^a and a $U(1)$ current $U(z)$. For more details see [34, 18]. The elliptic genera for these superconformal algebras present some interesting new features due to an unusual BPS bound. It turns out that the elliptic genus is *not* holomorphic (but still well-controlled) [18].

An important feature comes when our superconformal theory has a "factor" consisting of a free $N=2$ Gaussian multiplet, i.e. $(\phi(z),\psi(z))$ where ϕ is a complex boson and ψ is a complex fermion. (We are speaking loosely of a "chiral scalar." See the remarks above on the Gaussian model.) These arise when there are $U(1)$ symmetries in the superconformal field theory, thus extending the $\mathcal{N}=2$ superconformal algebra. In the D-brane context such symmetries are often associated with Wilson lines.

The R-sector partition function of $(\phi(z), \psi(z))$ in a given $U(1)$-charge sector is

$$q^{\frac{1}{2}p^2}\frac{\vartheta_1(z,\tau)}{\eta^3} \tag{3.74}$$

where p is the $U(1)$ charge. Note that this *vanishes* when we put $z=0$. Thus, in these cases the elliptic genus will vanish. One way of viewing this is that in the R sector the quantization of the Clifford algebra $\{\psi_0, \bar{\psi}_0\}=1$ leads to a doublet with $J_0=\pm 1/2$ so $\mathrm{Tr}e^{i\pi J_0}=0$. What we should do in this case is compute $\mathrm{Tr}J_0 e^{i\pi J_0}$. This will be nonzero. Indeed,

$$\frac{\partial}{\partial z}\vartheta_1|_{z=0} = -2\pi\eta^3 \tag{3.75}$$

and we see that the only contribution to the Witten index is $q^{\frac{1}{2}p^2}$, which typically cancels against a zero-point energy to give a constant. Similarly, when there are s $U(1)$ factors we should consider the modified elliptic genus $\mathrm{Tr}J_0^s e^{i\pi J_0}$.

Two important examples occur in counting BPS states for string theory compactification on T^5, which is associated with a contraction of the $\mathcal{A}_\gamma$ algebra, and in defining the elliptic genus of the $(0,4)$ MSW conformal field theory [9]. In both cases one must insert J_0^2.

4. Modularity, the elliptic genus, and polarity

4.1 Polar states and the elliptic genus

In general, the elliptic genus of a unitary $(2,2)$ superconformal field theory is a weak Jacobi form of weight zero and index m. Therefore, for the elliptic genus, the vector of forms $h_\mu(\tau)$ will have negative weight $w=-1/2$. As we have stressed, these are determined by their polar terms. This motivates the definition:

Definition A state in a representation of the $\mathcal{N}=2$ algebra which is an eigenstate of L_0 and J_0 is called a *polar state* if

$$p = 4m(L_0 - \frac{c}{24}) - J_0^2 = 4mL_0 - J_0^2 - m^2 < 0 \tag{4.1}$$

we refer to p as the *polarity* of the state.

Note that this notion of a polar state is spectral flow invariant.

For any nearly holomorphic Jacobi form it follows from (3.47) and (3.43) that the potential polar terms in the Fourier expansion of $h_\mu(\tau)$ are in one-one correspondence with the monomials $q^n y^\ell$ for which $4mn-\ell^2<0$ in accord with our definition of polar states.

Now let us count how many independent polar degeneracies $c(n,\ell)$ there are, subject to the constraints of spectral flow and unitarity, but not (yet) modular invariance. Unitarity shows that $c(n,\ell)=0$ for $n<0$. On the other hand, thanks to spectral flow, we can assume

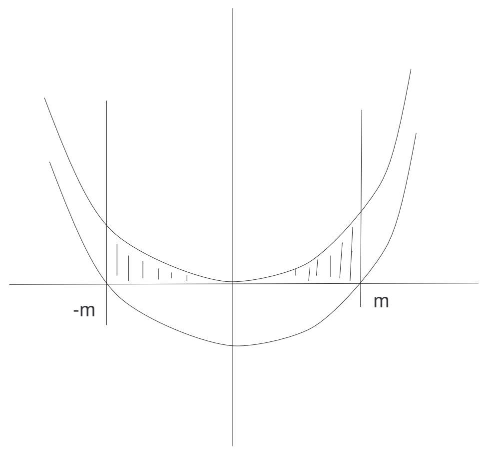

Figure 6: The shaded region contains the points (m, ℓ) in a fundamental domain for the action of spectral flow on the polar states. The region is divided in half if we include charge conjugation. There are order $\frac{m^2}{12} + \frac{5m}{8} + \mathcal{O}(m^{1/2})$ points in the fundamental domain for the action of charge conjugation and spectral flow, for large m.

that $-m \leq \ell \leq m$. In this case we can conclude that the nonvanishing polar degeneracies must correspond to values of (n, ℓ) such that $n \geq 0$ and

$$-m^2 \leq 4mn - \ell^2 < 0 \tag{4.2}$$

This defines the polar region shown in figure 6. Note carefully that the constant term $(n, \ell) = (0, 0)$ is *not* in the polar region.

It proves to be convenient to impose one simple condition of modular invariance, namely, invariance under $\gamma = -1$. This is charge conjugation invariance and it shows that $c(n, \ell) = (-1)^w c(n, -\ell)$ (with weight $w = 0$ for the elliptic genus) and therefore we will consider the independent degeneracies to be the coefficients of the monomials $q^n y^\ell$ with $1 \leq \ell \leq m$, $n \geq 0$ and (4.2). We can phrase this differently, spectral flow and charge conjugation generate an action of the infinite dihedral group $D_\infty = \mathbb{Z}_2 \rtimes \mathbb{Z}$ on the set of polar values of (n, ℓ). We are choosing a fundamental domain for the action of this group. Call this fundamental domain $\mathcal{P}$.

4.2 Constructing a form from its polar piece

We have repeatedly stressed that the polar part determines the entire elliptic genus, and more generally, the polar part of a vector valued nearly holomorphic modular form determines the entire form. This raises a general question: Can we construct a vector valued modular form with a specified weight, multiplier system, and polar part? This question is investigated in full generality in work with J. Manschot [25], section 4. In this section we give a simplified account, following closely the discussion of [25].

For simplicity let us suppose the multiplier system is trivial: $M(\gamma) = 1$, the weight w is a negative integer, and our desired modular form has the expansion:

$$f(\tau) = e^{-2\pi i \delta \tau} \sum_{n=0}^{\infty} \hat{f}(n) q^n \tag{4.3}$$

where δ is a positive integer. For example, physically $\delta = c/24$ and the weight $w = -d/2$ where d is the number of noncompact bosons. We are given the polar part

$$f^{-}(\tau) := \sum_{n-\delta<0} \hat{f}(n) e^{2\pi i(n-\delta)\tau} \tag{4.4}$$

and we wish to construct the full modular form $f(\tau)$.

It suffices to consider the case where $f^{-}(\tau) = e^{-2\pi i\delta\tau}$, since we can then take linear combinations of the result. The result we will now explain is that:

We can construct a modular form with polar piece $f^{-}(\tau) = e^{-2\pi i\delta\tau}$ iff and only if the associated cusp form

$$G^{(\delta)}(z) = \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{w-2}(-2\pi i\delta)^{1-w} e(\delta\gamma(z)). \tag{4.5}$$

vanishes.

This is perhaps a surprising statement. Indeed, one's first reaction is that it should be trivial to contruct the required form - why not just average over $SL(2,\mathbb{Z})$? Well, let us try:

Introduce

$$s_\gamma^{(\delta)}(\tau) := j(\gamma, \tau)^{-w} e(-\delta\gamma(\tau)).$$

Then it is elementary to check that

$$s_{\gamma\tilde{\gamma}}^{(\delta)}(\tau) = j(\tilde{\gamma}, \tau)^{-w} s_\gamma^{(\delta)}(\tilde{\gamma}\tau), \tag{4.6}$$

and hence $s_{\gamma\tilde{\gamma}}^{(\delta)}(\tau) = s_{\tilde{\gamma}}^{(\delta)}(\tau)$ for $\gamma \in \Gamma_\infty$. Accordingly, we attempt to average:

$$S^{(\delta)}(\tau) \stackrel{?}{=} \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} s_\gamma^{(\delta)}(\tau). \tag{4.7}$$

Formally, from Eq. (4.6) we find $S^{(\delta)}(\tilde{\gamma}\tau) = j(\tilde{\gamma}, \tau)^w S^{(\delta)}(\tau)$. Moreover, the cosets $[\pm 1]$ lead to the prescribed polar term and the remaining terms in the sum are regular for $\tau \to i\infty$. It would thus appear that we have succeeded, but in fact we have not. The problem with the naive attempt Eq. (4.7) is that for $c \to \infty$ we have $|s_\gamma^{(\delta)}(\tau)| \sim |c\tau|^{-w}$ and since we must have weight $w \le 0$, the series does not converge. We therefore must regularize the series.

To motivate our regularization let us suppose for the moment that $-w \in \mathbb{N}$. We use the identity

$$\gamma(\tau) = \frac{a}{c} - \frac{1}{c(c\tau + d)}, \tag{4.8}$$

which is valid for $c \neq 0$. This allows us to write

$$e(-\delta\gamma(\tau)) = e^{-2\pi i\delta\frac{a}{c}} e^{2\pi i\frac{\delta}{c(c\tau+d)}}. \tag{4.9}$$

An evident regularization would be to subtract the first $|w|$ terms from the Taylor series expansion of $e^{2\pi i \frac{\delta}{c(c\tau+d)}}$ around zero. Thus we introduce the regularized sum:

$$S^{(\delta)}_{\rm Reg}(\tau) := \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (s^{(\delta)}_\gamma(\tau) + t^{(\delta)}_\gamma(\tau)), \tag{4.10}$$

with $t^\delta_\gamma(\tau) := 0$ for $c = 0$ and

$$t^{(\delta)}_\gamma(\tau) := -j(\gamma,\tau)^{-w} e^{-2\pi i\delta\frac{a}{c}} \sum_{j=0}^{|w|} \frac{1}{j!}\left(\frac{1}{c(c\tau+d)}\right)^j (2\pi i\delta)^j\,. \tag{4.11}$$

for $c \neq 0$.

Our regularization has rendered the sum convergent, but now we have spoiled manifest covariance under modular transformations! But the situation is not as bad as it might appear at first, thanks to the theory of periods.

♣ FIGURE OF PERIOD CYCLE. ♣

For any function $h(\tau)$ on $\mathcal{H}$ decaying sufficiently rapidly at $\mathrm{Im}(\tau)\to\infty$ we can define its period function

$$p(\tau,\bar{y},\bar{h}) := \frac{1}{\Gamma(1-w)}\int_{\bar{y}}^{-i\infty} \overline{h(z)}(\bar{z}-\tau)^{-w} d\bar{z}. \tag{4.12}$$

The contour of z is shown in FIGUREa. When projected to the modular surface $\mathcal{H}/PSL(2,\mathbb{Z})$ it is a period on the noncompact cycle in FIGUREb.

Then we claim that

$$t^{(\delta)}_\gamma(\tau) = p(\tau,\gamma^{-1}(-i\infty),\overline{g^{(\delta)}_\gamma}), \tag{4.13}$$

where

$$g^{(\delta)}_\gamma(z) := j(\gamma,z)^{w-2}(-2\pi i\delta)^{1-w} e(\delta\gamma(z)). \tag{4.14}$$

Note that $\gamma^{-1}(-i\infty) = a/c$.

Now, $g^{(\delta)}_\gamma(z)$ transforms simply and this allows us to write a useful formula for the transformation of $t^{(\delta)}_\gamma(\tau)$. When we make a transformation $\tau\to\tilde{\gamma}\tau$ we must change variables $z\to\tilde{\gamma}z$ in the period integral. This shifts the domain of integration, and we get:

$$t^{(\delta)}_\gamma(\tilde{\gamma}\tau) = j(\tilde{\gamma},\tau)^w\left[t^{(\delta)}_{\gamma\tilde{\gamma}}(\tau) - p(\tau,\tilde{\gamma}^{-1}(-i\infty),\overline{g^{(\delta)}_{\gamma\tilde{\gamma}}})\right]. \tag{4.15}$$

The net effect of this shift is that we actually have the transformation law:[8]

$$\hat{S}^{(\delta)}_{\rm Reg}(\tilde{\gamma}\tau) = j(\tilde{\gamma},\tau)^w\left[\hat{S}^{(\delta)}_{\rm Reg}(\tau) - p^{(\delta)}(\tau,\tilde{\gamma})\right], \tag{4.16}$$

where

$$p^{(\delta)}(\tau,\tilde{\gamma}) := p(\tau,\tilde{\gamma}^{-1}(\infty),\overline{G^{(\delta)}}) = \int_{-\tilde{d}/\tilde{c}}^{-i\infty} \overline{G^{(\delta)}(z)}(\bar{z}-\tau)^{-w} d\bar{z}, \tag{4.17}$$

[8] We are skipping over several technicalities at this point, which are addressed in [25]. The net result is that in general one must add a constant to $S^{(\delta)}_{\rm Reg}$ to get a good transformation law. This is the reason for the notation $\hat{S}^{(\delta)}_{\rm Reg}$.

is the period our cusp form defined in (4.5). The function $p^{(\delta)}$ is an obstruction to the existence of $f(\tau)$, and it can be shown that it vanishes iff $G^{(\delta)}(\tau)$ vanishes.

In contrast to Eq. (4.7), the series (4.5) for $G^{(\delta)}(z)$ is nicely convergent. It therefore follows that $G^{(\delta)}(\tau)$ is a vector-valued modular form of weight $2-w$ transforming according to

$$G^{(\delta)}(\gamma\tau) = j(\gamma,\tau)^{2-w} G^{(\delta)}(\tau). \tag{4.18}$$

In fact, $G^{(\delta)}$ is a *cusp form*, that is, the components vanish for $\tau \to i\infty \cup \Gamma(i\infty)$. This follows since it is clear from the series expansion that $G^{(\delta)}$ vanishes for $\tau \to i\infty$.

Evidently, the polar parts of negative weight forms are very special. They are of the form $\sum_{\delta>0} a_\delta q^{-\delta}$ so that

$$\sum a_\delta G^{(\delta)}(\tau) = 0 \tag{4.19}$$

Remarks

1. Note that $g_\gamma^{(\delta)}(z)$ it is just like $s_\gamma^{(\delta)}(z)$ with the replacement $w \to 2-w$, $\delta \to -\delta$ and a different prefactor.

2. ♣ Comment on modular anomaly analogy. Also analogy to the Mittag-Leffler problem. ♣

3. In the present case there is another approach using Bol's identity. ♣ EXPLAIN ♣

4. It is, unfortunately, rather difficult to tell when the cusp form $G^{(\delta)}$ vanishes. Let us take, for example the case of weight $w = -10$. Then the cusp form is proportional to

$$\Delta(\tau) = \sum_{n=0}^{\infty} \tau(n) q^n \tag{4.20}$$

   In this case $G^{(\delta)}$ vanishes iff $\tau(\delta) = 0$. But deciding when $\tau(n)$ vanishes is considered a very deep problem in number theory.

5. When applied to the elliptic genus we have

$$0 \to \tilde{J}_{0,m} \overset{\mathrm{Pol}}{\to} V_m \to S_{5/2}(\Gamma, M) \tag{4.21}$$

   where M is the multiplier system contragredient to that of the level m theta functions. It can be shown that the rightmost arrow is not onto, and it is of interest to understand the image of this map better.

4.3 The Rademacher expansion

In the case when the cuspform obstruction (4.19) does vanish the Poincaré series leads to a very nice formula for the Fourier coefficients of the nonpolar part of the modular form. These are the degeneracies of physical interest in black hole physics.

From the Fourier expansion we know that

$$\hat{f}(m) = \int_{\tau}^{\tau+1} e^{-2\pi i(n-\delta)\tau} f(\tau) \tag{4.22}$$

We now substitute the Poincaré series and integrate term by term. For $m - \delta > 0$ the contributions of $t_\gamma^{(\delta)}$ vanish, and the integral of $s_\gamma^{(\delta)}$ is a Bessel function. In this way one can derive the Rademacher expansion:

$$\begin{aligned}\hat{f}(m) = 2\pi \sum_{n-\delta<0} \hat{f}(n) \sum_{c=1}^{\infty} \frac{1}{c} K_c(m-\delta, n-\delta) \\ \times \left(\frac{|n-\delta|}{m-\delta}\right)^{(1-w)/2} I_{1-w}\left(\frac{4\pi}{c}\sqrt{(m-\delta)|n-\delta|}\right),\end{aligned} \tag{4.23}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind. $I_\nu(z)$ is given as an infinite sum by

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)}, \tag{4.24}$$

and $K_c(m-\delta, n-\delta)$ is known as a *Kloosterman sum*:

$$K_c(m-\delta, n-\delta) := i^{-w} \sum_{\substack{-c\le d<0 \\ (c,d)=1}} e\left((n-\delta)\frac{a}{c} + (m-\delta)\frac{d}{c}\right), \tag{4.25}$$

One simple consequence of (4.23) is the asymptotic behavior of the Fourier coefficients. The Bessel function has asymptotics

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^x \qquad \mathrm{Re}(x) \to +\infty \tag{4.26}$$

Therefore, the leading term is the $c = 1$ term in the infinite sum in (4.23). We get the asymptotics:

$$\hat{f}(m) \sim \frac{1}{\sqrt{2}} \hat{f}(0) \delta^{1/4-w/2} (m-\delta)^{w/2-3/4} e^{4\pi\sqrt{\delta(m-\delta)}} \tag{4.27}$$

Remarks

1. This is sometimes referred to as the Hardy-Ramanujam formula in the math literature. In the physical context with $\delta = c/24$ the asymptotic formula

$$\log \hat{f}(m) \sim 2\pi\sqrt{\frac{c}{6}m} + \mathcal{O}(\log m) \tag{4.28}$$

   is often referred to as Cardy's formula in the physics literature.

2. It is quite important that the asymptotics (4.27) are only valid for $m - \delta \gg 1$, and indeed the $m - \delta$ in the argument of the Bessel functions should warn us that we should only use (4.28) when $m \gg \delta$. In the physical applications, the case $\delta \gg 1$ and $m - \delta \sim \mathcal{O}(1)$ turns out to be quite relevant as well, and in this regime the asymptotics is rather different. For example, if we define

$$\eta^{-\chi}(\tau) = q^{-\chi/24} \sum_{n=0}^{\infty} p_\chi(n) q^n. \tag{4.29}$$

then [25]

$$p_\chi\left(\frac{\chi}{24}+\ell\right) \sim_{\chi\to\infty} const.\chi^{-1/2}\exp\left(\frac{\chi}{2}\left(1+\log\frac{\pi}{6}\right)+\frac{\pi^2}{3}\ell\right). \tag{4.30}$$

3. The Rademacher expansion (4.23) can be generalized to arbitrary weight $w \le 0$ and multiplier system. The result is

$$\begin{aligned}\hat{f}_\mu(m) = 2\pi \sum_{n-\Delta_\nu<0} \hat{f}_\nu(n) \sum_{c=1}^{\infty} \frac{1}{c} K_c(m-\Delta_\mu, n-\Delta_\nu) \\ \times \left(\frac{|n-\Delta_\nu|}{m-\Delta_\mu}\right)^{(1-w)/2} I_{1-w}\left(\frac{4\pi}{c}\sqrt{(m-\Delta_\mu)|n-\Delta_\nu|}\right),\end{aligned} \tag{4.31}$$

where we now have a generalized Kloosterman sum:

$$K_c(m-\Delta_\mu, n-\Delta_\nu) := i^{-w} \sum_{\substack{-c\le d<0\\(c,d)=1}} M^{-1}(\gamma)_\mu^{\ \nu} e\left((n-\Delta_\nu)\frac{a}{c}+(m-\Delta_\mu)\frac{d}{c}\right), \tag{4.32}$$

For a discussion and derivation in the physics literature see [11], appendix B. The proof makes use of Rademacher's fiendishly clever contour deformation.

4.4 AdS/CFT and the Fareytail expansion

We will be brief here since this is extensively covered in [11, 26].

♣ Brief on this:

1. Regularized sum for elliptic genus.

2. Interpret sum over $(c, d) = 1$ as a sum over BTZ geometries, hence an example of AdS/CFT.

3. Most important point to convey is that the cosmic censorship bound for black hole solutions without naked singularity is $4m(L_0-c/24)-J_0^2 \ge 0$. This coincides with positive polarity of the states. (Be careful that negative polarity states can still correspond to smooth supergravity solutions – Lunin+Maldacena.)

♣

4.5 Three-dimensional gravity

In an intriguing paper Witten has revived the study of 3d quantum gravity [40]. Twenty years ago it was observed [1, 38, 2] that the classical 2+1 dimensional (super) gravity actions could be written as Chern-Simons gauge theories, thus raising hopes of a complete solution of quantum gravity in 2+1 dimensions. We will restrict attention to the case of negative cosmological constant. (See [40] for a thorough discussion of why one should do this.)

Let G denote the 3-dimensional Newton constant, and ℓ the AdS radius. Then we can consider the action

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{g}(\mathcal{R}+\frac{2}{\ell^2}) + \frac{k}{4\pi}\int \mathrm{Tr}\left(\omega d\omega+\frac{2}{3}\omega^3\right) \tag{4.33}$$

where ω is the $SO(2,1)$ spin connection. k is suitably quantized due to the topology of $SO(2,1)$. (For a detailed and careful discussion see [40].)

By making the change of variables:

$$A_- = \omega - *e/\ell \qquad A_+ = \omega + *e/\ell \tag{4.34}$$

$$k_- = \frac{\ell}{16G} + \frac{k}{2} \qquad k_+ = \frac{\ell}{16G} - \frac{k}{2} \tag{4.35}$$

where $*e_{ab} = \epsilon_{abc}e^c$, and e^c is the dreibein, one can write a *classically equivalent* action

$$S = \frac{k_-}{4\pi}\int \mathrm{Tr}\left(A_- dA_- + \frac{2}{3}A_-^3\right) - \frac{k_+}{4\pi}\int \mathrm{Tr}\left(A_+ dA_+ + \frac{2}{3}A_+^3\right) \tag{4.36}$$

These classical equivalences suggest that one might be able to formulate 2+1 dimensional anti-deSitter gravity as a gauge theory. However, there is an important conceptual difficulty in doing so. In gauge theory there is no requirement that the dreibein e^a correspond to an invertible metric. In order to see how such a condition can drastically change the physics consider the case of gravity in $0+1$ dimensions. Here the propagator can be written as

$$\int de e^{-\int e(p^2+m^2)} \tag{4.37}$$

♣ BE MORE PRECISE HERE! ♣ For invertible dreibeins we integrate e from 0 to ∞ to produce a Laplace transform. This gives us the propagator. If we instead ignore invertibility we obtain a Fourier transform $\delta(p^2+m^2)$, with the wrong physics. [9]

Nevertheless, Witten takes (4.36) as an indication that the gravity partition function holomorphically factorizes. This is perhaps best justified in the case when $k_+ = 0$ (but see the remark below). Let us just accept this and see how to apply the above techniques.

We have seen above that for $w = 0$ and trivial multiplier system one can construct a holomorphic candidate partition function with arbitrary polar part.

♣ No local modes. Brown-Henneaux singleton modes at infinity are Virasoro descendents. ♣

Witten defines "pure gravity" to be a theory with a spectrum as close as possible to having a vacuum and only Virasoro descendents. The vacuum character for $c = 24k$ is

$$\chi_v^{(k)} = q^{-k}\prod_{n=2}^{\infty}\frac{1}{1-q^n} = q^{-k+1/24}(1-q)\eta(\tau)^{-1} \tag{4.38}$$

Note there is no factor of $\frac{1}{1-q}$ in the first product because $L_{-1}|0\rangle$ is a null state, and it generates all the other null states in the Verma module. From the second equality it is clear that the expression is far from modular.

Because of AdS/CFT duality, it is reasonable to insist that a quantum theory of pure gravity should have a modular invariant partition function.

From our discussion above, we know that we can prescribe the polar part arbitrarily, but once we have done so, the nonpolar terms, beginning at $\mathcal{O}(q)$ are determined. Witten

[9] I learned this remark from N. Seiberg.

interprets the nondescendent contributions to $\mathcal{O}(q)$ as arising from BTZ black holes. This fits in well with the interpretation of polarity in the Farey tail story.

These considerations motivate the definition of an "extremal conformal field theory" to be a conformal field theory with holomorphic partition function all of whose polar terms are given by Virasoro descendents. That is the partition function must have the form

$$Z_k = \chi_v^{(k)}(\tau) + \text{nonpolar} \tag{4.39}$$

Witten's hypothesis was that the partition function of pure $2+1$ dimensional quantum gravity with $k = k_L = k_R$ is $|Z_k|^2$.

Remarks

1. Since 3D gravity is not our principle theme we have glossed over a number of important subtleties in our discussion above. These include distinctions between the first and second order formalism and the appearance of an extra gauge invariance when $k_+ = 0$ which ruins the relation of ω to the spin connection. For a more careful account see [Witten, Strominger-chiral gravity].

2. Extremal conformal field theories are not easily constructed. ♣ REVIEW STATUS OF CONSTRUCTION.

5. Extremal $\mathcal{N} = 2$ superconformal field theories

We will now describe a new application of the constraints of modularity of the elliptic genus of $\mathcal{N} = 2$ superconformal field theories.

This section summarizes work done with M. Gaberdiel, S. Gukov, C. Keller, H. Ooguri, and C. Vafa which will appear in [15].

There are two motivations for this new work:

- In recent years many new models of flux+orientifold compactifications of type II string theories have been studied in which, it is alleged, all the moduli of the theory are fixed. Some of these models preserve AdS supersymmetry. In the context of M-theory, analogous compactifications on CY 4-folds with flux lead to a large number of compactifications in which the moduli are fixed. If the Kaluza-Klein length scale is small, and the AdS cosmological constant is small (so that the AdS length scale is large) then one expects the spectrum to approach that of "pure $\mathcal{N} = 2$ supergravity."

  ♣ IN VIEW OF FREDERIK'S REMARKS THIS MIGHT BE COMPLETELY WRONG ♣

- Supergravities with (p, q) supersymmetry also have Chern-Simons actions [1, 2]. It is therefore natural to try to extend Witten's work to these theories.

5.1 Computing the number of potentially polar states

****

♣ NEED NEW SEGUE ♣

****

These monomials $q^n y^\ell$ in the fundamental domain of D_∞ span a vector space V_m of dimension:

$$P(m) = \sum_{r=1}^{m} \lceil \frac{r^2}{4m} \rceil \tag{5.1}$$

Note that we want the *ceiling* function and not the floor function. We include $n = 0$ up to the largest n with $4nm - r^2 < 0$ for each $r = 1, \ldots, m$. One can show [15] (following results in [12]) that

$$P(m) = \frac{m^2}{12} + \frac{5m}{8} + A(m) \tag{5.2}$$

where $A(m)$ is the arithmetic function which, *roughly speaking* grows like $m^{1/2}$. We sketch how this comes about in the remarks below.

On the other hand, we can use the above structure theorem to compute the dimension of the space of weak Jacobi forms of weight zero and index m. A natural basis of $\tilde{J}_{0,m}$ is given by

$$(\tilde{\phi}_{-2,1})^a (\tilde{\phi}_{0,1})^b E_4^c E_6^d \tag{5.3}$$

where $a, b, c, d \geq 0$ integral, $a + b = m$, and $a = 2c + 3d$. Therefore

$$j(m) := \dim \tilde{J}_{0,m} = \sum_{a=0}^{2m} \dim M_a(SL(2,\mathbb{Z})) \tag{5.4}$$

Using the above results on $\dim M_w$ it is straightforward to show that

$$j(m) = \dim \tilde{J}_{0,m} = \frac{m^2}{12} + \frac{m}{2} + \left(\delta_{s,0} + \frac{s}{2} - \frac{s^2}{12} \right) \tag{5.5}$$

where $m = 6\rho + s$ with $\rho \geq 0$ and $0 \leq s \leq 5$.

$$\dim \tilde{J}_{0,m} = \begin{cases} m^2/12 + m/2 + 1 & m = 0 \bmod 6 \\ m^2/12 + m/2 + 5/12 & m = 1 \bmod 6 \\ m^2/12 + m/2 + 2/3 & m = 2 \bmod 6 \\ m^2/12 + m/2 + 3/4 & m = 3 \bmod 6 \\ m^2/12 + m/2 + 2/3 & m = 4 \bmod 6 \\ m^2/12 + m/2 + 5/12 & m = 5 \bmod 6 \end{cases} \tag{5.6}$$

We can tabulate the first few values

m	$\dim \tilde{J}_{0,m}$	$\dim V_m$
$m=0$	1	0
$m=1$	1	1
$m=2$	2	2
$m=3$	3	3
$m=4$	4	4
$m=5$	5	6
$m=6$	7	8
$m=7$	8	9
$m=8$	10	11
$m=9$	12	13
$m=10$	14	16
$m=11$	16	18
$m=12$	19	21

(5.7)

We thus learn that $j(m) < P(m)$ in general. Indeed, for large m, $P(m) - j(m)$ is of order $\frac{m}{8}$. Thus, the constraints of modular invariance impose further conditions on the polar terms. That is, one cannot specify arbitrarily the polar terms in a vector of negative weight forms!

For our discussion below it will be useful to rephrase this as follows. Given a weak Jacobi form ϕ we can define its *polar polynomial* as

$$\mathrm{Pol}(\phi) := \sum_{4mn-\ell^2<0, 1\leq\ell\leq m} c(n,\ell) q^n y^\ell \tag{5.8}$$

Then we have an injective map

$$0 \to \tilde{J}_{0,m} \stackrel{\mathrm{Pol}}{\to} V_m \tag{5.9}$$

which is *not* onto.

Remarks

1. We prove the assertion above about $P(m)$ as follows. Write

$$\sum_{r=1}^{m} \lceil \frac{r^2}{4m} \rceil = \sum_{r=1}^{m} \frac{r^2}{4m} - \sum_{r=1}^{m} ((\frac{r^2}{4m})) + \frac{1}{2} \sum_{r=1}^{m} \left(\lceil \frac{r^2}{4m} \rceil - \lfloor \frac{r^2}{4m} \rfloor \right) \tag{5.10}$$

where, as in [12],

$$((x)) := x - \frac{1}{2}(\lceil x \rceil + \lfloor x \rfloor) = \begin{cases} 0 & x \in \mathbb{Z} \\ \alpha - \frac{1}{2} & x = n + \alpha, 0 < \alpha < 1 \end{cases}$$

Note that $((x))$ is the sawtooth function. It is periodic of period 1. Now, the main term comes from the elementary formula

$$\sum_{r=1}^{m}\frac{r^2}{4m}=\frac{m^2}{12}+\frac{m}{8}+\frac{1}{24} \tag{5.11}$$

Next, note that in the last sum in (5.10) each summand is 1 unless $r^2/4m$ is an integer. It turns out that

$$\sum_{r=1}^{m}\lceil\frac{r^2}{4m}\rceil-\sum_{r=1}^{m}\lfloor\frac{r^2}{4m}\rfloor=m-\lfloor\frac{b}{2}\rfloor \tag{5.12}$$

where b is the largest integer such that b^2 divides m. It is thus of order $m^{1/2}$. Finally we come to the most subtle term $\sum_{r=1}^{m}((\frac{r^2}{4m}))$. The numbers $((\frac{r^2}{4m}))$ are, very roughly speaking, randomly distributed between $-1/2$ and $+1/2$. So we sum over a random walk and expect the sum to grow like $m^{1/2}$. One can give an exact formula for this term in terms of class numbers.

It is up to us to define what we mean by "pure $\mathcal{N}=2$ supergravity." We will try to define it by taking the NS partition function to be a close as possible to the character of the irreducible $\mathcal{N}=2$ representation built on the vacuum.

To write down this character, the Verma module has character given by

$$\chi_{Verma}=q^{-m/4}\frac{\prod_{m=0}^{\infty}(1+yq^{m+1/2})(1+y^{-1}q^{m+1/2})}{\prod_{m=1}^{\infty}(1-q^m)^2} \tag{5.13}$$

However, we must mod out by the sub-Verma module for the subalgebra generated by $G^{\pm}_{-1/2}, L_{-1}$ acting on the vacuum, as these generate all the null vectors in the module [6]. That submodule has character

$$\frac{(1+yq^{1/2})(1+y^{-1}q^{1/2})}{(1-q)} \tag{5.14}$$

Taking the quotient we get the character of the irreducible vacuum representation:

$$\chi_v=q^{-m/4}(1-q)\frac{\prod_{m=0}^{\infty}(1+yq^{m+3/2})(1+y^{-1}q^{m+3/2})}{\prod_{m=1}^{\infty}(1-q^m)^2} \tag{5.15}$$

We assume here $\hat{c}>1$ so that the only null states are from

We adopt the

Definition: "Pure $\mathcal{N}=(2,2)$ supergravity" is the hypothetical theory whose partition satisfies:

$$Z_{NS}=|\sum_{s\in\mathbb{Z}}SF_s\chi_v|^2+\sum_{\text{Nonpolar}}a(n,\ell;\tilde{n},\tilde{\ell})q^n y^\ell \bar{q}^{\tilde{n}}\bar{y}^{\tilde{\ell}} \tag{5.16}$$

for some positive integer coefficients $a(n,\ell;\tilde{n},\tilde{\ell})$ so that Z_{NS} is left- and right spectral flow invariant for all integral s, and the sum over nonpolar states means that either the left-moving or the right-moving state has nonnegative polarity.

With this definition we can compute $Z_{RR} = SF_{1/2}\widetilde{SF}_{1/2}Z_{NS}$ and then, specializing to $z \to z + \frac{1}{2}$ and $\bar{z} = \frac{1}{2}$ we can compute

$$\chi(\tau, z) = 2(-1)^m \sum_{\theta \in \mathbb{Z} + \frac{1}{2}} SF_\theta \chi_v + \text{Nonpolar} \tag{5.17}$$

As a test of the existence of pure $\mathcal{N} = 2$ supergravity we would therefore like to construct a weak Jacobi form of weight zero and index m with integral Fourier coefficients, which coincides with (5.17). We therefore examine the polar polynomial of (5.17). It can be shown that this is given by

$$p^m_{\text{sugra}} := \text{Pol}(SF_{1/2}\chi_v) \tag{5.18}$$

Let ϕ_i denote a basis of $\tilde{J}_{0,m}$. We search for a solution to

$$\sum_{i=1}^{j(m)} x_i \text{Pol}(\phi_i) = p^m_{\text{sugra}} \tag{5.19}$$

This is an explicit linear equation one can attempt to analyze. To write it explicitly we will order the basis monomials $q^{n(a)}y^{l(a)}$ where $a = 1, \dots, \dim V_m = P(m)$ so that polarity increases as a increases, and terms with the same polarity are ordered in increasing powers of y. For example, an ordered basis for V_5 would be

$$y^5, y^4, y^3, qy^5, y^2, y^1 \tag{5.20}$$

with $a = 1, \dots, 6$.

Choosing a basis ϕ_i for $\tilde{J}_{0,m}$ we can define a matrix N_{ia} of dimensions $j(m) \times P(m)$ from the expansion

$$\text{Pol}(\phi_i(q, y)) = \sum_{a=1}^{P(m)} N_{ia} q^{n(a)} y^{\ell(a)} \tag{5.21}$$

Defining coefficients d_a by

$$p^m_{\text{sugra}} = \sum_{a=1}^{P(m)} d_a q^{n(a)} y^{\ell(a)} \tag{5.22}$$

we are trying to solve the linear equation

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a, \qquad\qquad a = 1, \dots, P(m) \tag{5.23}$$

In [15] it is shown that

- There are only solutions for $1 \leq m \leq 5$ and $m = 7, 8, 11, 13$, but *there is no solution for $m = 6, 9, 10, 12$ and $14 \leq m \leq 36$.* Note that since $P(m) > j(m)$ for $m \geq 5$ the existence of solutions for $m = 5, 7, 8, 11, 13$ requires a "miracle" since in (5.24) there are more equations than unknowns.

- Moreover, although the solutions x_i are complicated rational numbers the Fourier expansion coefficients $c(n,\ell)$ appear to be integral.

Evidently, "pure $\mathcal{N}=2$ supergravity" in the strict sense we have defined it cannot exist. But it is natural to define "nearly pure supergravity" by trying to match only the terms of most negative polarity. Indeed, it turns out that, for $1 \leq m \leq 36$, the system of equations

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a, \qquad a = 1, \ldots, j(m) \tag{5.24}$$

does have solutions and leads to a Jacobi form $\sum x_i \phi_i$ with integral Fourier coefficients, *except* for $m = 17$.

Let $P_\beta(m)$ be the number of independent polar monomials of polarity $\leq -\beta$, and let β_* be the *smallest* integer β such that

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a, \qquad a = 1, \ldots, P_\beta(m) \tag{5.25}$$

admits a solution x_i such that $\sum x_i \phi_i$ has *integral* coefficients in its Fourier expansion.

Numerical and analytic evidence in [15] strongly suggests that $P_\beta(m) \leq j(m)$. Using an estimate analogous to (5.2) one finds that

$$\beta_* \geq \frac{m}{2} + \mathcal{O}(m^{1/2}) \tag{5.26}$$

for large m.

Now, a monomial $q^n y^\ell$ of polarity $-\beta$ corresponds by spectral flow to a state in the NS sector with

$$L_0^{NS} = \frac{m}{4} + \frac{(J_0^{NS})^2}{4m} - \frac{\beta}{4m} \tag{5.27}$$

Therefore, if we accept (5.26) then we can obtain an interesting constraint on the spectrum of a $(2,2)$ AdS3 supergravity with a holographically dual CFT: It must contain at least one state which is a left-moving $\mathcal{N}=2$ primary (not necessarily chiral primary) tensored with a right-moving chiral primary such that

$$h^{NS} > \frac{m}{4} + \frac{(J_0^{NS})^2}{4m} - \frac{1}{8} + \mathcal{O}(m^{-1/2}) \tag{5.28}$$

In the other direction, it is also shown in [15] that one can construct candidate Jacobi forms with integral coefficients which agree with the $\mathcal{N}=2$ vacuum character up to $h^{NS} \leq \frac{5m}{16}$.

Thus, we reach some open problems:

1. Does (5.28) place interesting constraints on flux compactifications?
2. Can one construct "near extremal" $\mathcal{N}=2$ theories whose elliptic genus matches that of pure supergravity up to $h^{NS} = \frac{5m}{16}$?

6. BPS Wallcrossing from supergravity

6.1 What are BPS states and why do we care about them ?

The "space of BPS states" has been a very useful concept, central to many of the key advances which have emerged in string theory and supersymmetric field theory in the past 30 years. In this lecture we will be describing recent progress in understanding some phenomena associated with the question of stability of BPS states as a function of parameters describing the space of vacua.

6.1.1 Defining the space of BPS states

Generally speaking, spaces of BPS states are associated to string/field theories on asymptotically AdS or Minkowskian spacetime, M_n, with some unbroken extended supersymmetry.

For definiteness we focus on theories in (asymptotically) 4-dimensional Minkowski space with $\mathcal{N}=2$ supersymmetry. The Poincaré superalgebra has even generators $\mathbb{C}\oplus\mathbb{R}^4\oplus\Lambda^2\mathbb{R}^4$ with basis $\hat{Z},\hat{P}_\mu,\hat{M}_{\mu\nu}$ and odd generators $S\otimes\mathbb{C}^2$, with basis $Q_{i\alpha}$. Here α is a chiral spinor index, and $i=1,2$ is an R-symmetry index. The commutation relation on the odd generators is

$$\{Q_{i\alpha},Q_{j\beta}\}=\delta_{ij}(C\Gamma^\mu)_{\alpha\beta}P_\mu+\epsilon_{ij}C_{\alpha\beta}\hat{Z} \tag{6.1}$$

The operator $\hat{Z}$ is central.

♣ GIVE FULL SUPERALGEBRA? Say something about C,Γ^μ. Ref to Randjbar-Daemi review. ♣

Now, in the theory there is a *one-particle Hilbert space* of states $\mathcal{H}$, and since $\hat{Z}$ is central we can decompose this Hilbert space into *isotypical components*, i.e. eignspaces of $\hat{Z}$:

$$\mathcal{H}=\oplus_{z\in\mathbb{C}}\mathcal{H}_{\hat{Z}=z} \tag{6.2}$$

Now a key Lemma states that on the space $\mathcal{H}_{\hat{Z}=z}$ the eigenvalues E of the Hamiltonian is bounded below by $|z|$. To prove this we view (6.1) as a 6-dimensional supersymmetry algebra

$$\{Q_A,Q_B\}=(C\Gamma^M)_{AB}P_M \tag{6.3}$$

where $\hat{P}_M=(\hat{P}_0,\vec{P},\widehat{P_4},\hat{P}_5)$ is a 6-dimensional momentum, and $\hat{P}_4+i\hat{P}_5=\hat{Z}$. Now for physically sensible representations where we have diagonalized $\hat{P}_M$ we have

$$M_{6D}^2=E^2-\vec{P}^2-|z|^2\geq 0 \tag{6.4}$$

Definition: The space of BPS states, denoted $\mathcal{H}_{BPS}$ is the subspace of $\mathcal{H}$ on which the BPS bound $E=|z|$ is saturated.

Note that the bound is only saturated for $\vec{P}=0$, that is, for particles at rest, and moreover $M_{6D}=0$ implies these are "small representations" of supersymmetry, that is, some combination of supercharges $\sum c_AQ_A$ annihilates the representation of the little group. As stressed by Seiberg and Witten, these representations are more rigid, and hence their degeneracies are more computable than generic representations.

6.1.2 Type II string theories

Now let us specialize to type II string theories on $M_4 \times X$. Since M_4 is noncompact, to define the $\mathcal{N}=2$ superalgebra on the Hilbert space of the theory we must specify boundary conditions for the massless fields. We denote these collectively as

$$\Phi_\infty = \lim_{\vec{x}\to\infty} (g_{\mu\nu}, \phi, B_{\mu\nu}, G) \tag{6.5}$$

for the metric, dilaton, B-field, and RR field, respectively. In particular the 1-particle Hilbert space, as a representation of $\mathcal{N}=2$, depends on these boundary conditions, so we denote it by $\mathcal{H}_{\Phi_\infty}$.

In these compactifications there is typically an unbroken abelian gauge group (coming from the RR fields) and it turns out that $\mathcal{H}_{\Phi_\infty}$ is graded by superselection, or charge sectors. The charge group is the twisted K-theory of X:

$$\mathcal{H}_{\Phi_\infty} = \oplus_\Gamma \mathcal{H}^\Gamma_{\Phi_\infty} \tag{6.6}$$

where the sum runs over elements in some twisted K-theory of X. [10] An important element in what follows is that the group of charges (modulo torsion) has a *symplectic product*. For the case of $K(X)$ this symplectic product is provided by the index theorem. ♣ EXPLAIN MORE HERE? ♣

Now we put these two ideas together: Consider type IIA strings on a static Calabi-Yau 3-fold X with flat B-field and flat RR fields at infinity. Under these circumstances the unbroken supersymmetry is $d=4, \mathcal{N}=2$. Therefore, each subspace $\mathcal{H}^\Gamma_{\Phi_\infty}$ is a representation of $\mathcal{N}=2$ where $\hat{Z}$ is proportional to the identity, so $\hat{Z} = Z(\Gamma; \Phi_\infty)$. This defines the crucial central charge function. We can now study the BPS spectrum

$$\mathcal{H}_{BPS} = \oplus_{\Gamma\in K^0(X)} \mathcal{H}^\Gamma_{\Phi_\infty, BPS} \tag{6.7}$$

In all known examples the spaces $\mathcal{H}^\Gamma_{\Phi_\infty, BPS}$ are finite dimensional, as strongly expected on physical grounds. ♣ SAY MORE ? ♣

6.1.3 Dependence on moduli

The space of boundary values Φ_∞ preserving the $d=4, \mathcal{N}=2$ supersymmetry will be referred to as the *moduli space of vacua*. The spaces $\mathcal{H}^\Gamma_{\Phi_\infty, BPS}$ are locally constant but *not* globally constant on moduli space.

The moduli space splits (at least locally) as a product $\mathcal{M}_{HM} \times \mathcal{M}_{VM}$ of hypermultiplet and vectormultiplet moduli. To fix ideas let us take the IIA theory. Then the hypermultiplet moduli consist of the complex structure, dilaton, and RR fields. The vectormultiplet moduli are given by the complexified Kahler class. At large radius this is represented by

[10] To be more precise, the charge runs over $K^{\tau+d}(X)$ where τ is a twisting whose isomorphism class is determined by the topological class of the B-field, and $d=0,1$ for IIB, IIA theory, respectively. For orbifolds we use twisted equivariant K-theory, and for orientifolds a certain hybrid of twisted and equivariant KR theory which will be described in forthcoming work with J. Distler and D. Freed. Incidentally, the assertion (6.6) is not true for flux sectors [13, 14].

$t = B + iJ$. Throughout these lectures we work in the large radius regime, and therefore we regard t as taking values in a kind of upper half plane

$$t \in H^2(X, \mathbb{R}) \oplus i\mathcal{K} \tag{6.8}$$

where $\mathcal{K}$ is the Kähler cone of X.

It has been known for at least 12 years [Ref: Harvey-Moore, 1995] that $\mathcal{H}^{\Gamma}_{\Phi_\infty,BPS}$ *can* change discontinuously as a function of hypermultiplet moduli. In all known examples this happens when a vectormultiplet and hypermultiplet representation simultaneously becomes massless. Therefore, to find a computable measure of the number of BPS states we should introduce an index. For the $d = 4, \mathcal{N} = 2$ algebra the appropriate index is the second helicity supertrace:

$$\Omega(\Gamma; \Phi_\infty) := -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}^{\Gamma}_{\Phi_\infty,BPS}}(2J_3)^2(-1)^{2J_3} \tag{6.9}$$

♣ SHOW THAT VM and HM contribute with opposite signs? ♣

In the known examples the locus where

1. What are BPS states and why do we care about them?

   c.) Focus on type II strings on $M_4 \times X$ and $d = 4$ susy gauge theory. Common feature: Moduli space of vacua. Abelian gauge theory. Symplectic product on the lattice of charges.

   d.) Moduli. Dependence of $\mathcal{H}_{BPS}$ on moduli. HM moduli and VM moduli. Helicity supertraces. Define $\Omega(\Gamma; t)$ and Poincaré polynomials $\Omega(\Gamma; x, y; t)$.

2. Physical motivations

   a. Review Strominger-Vafa.

   b. Topological string theory.

   c. Quantum corrections, automorphic forms, BPS algebras.

3. Mathematical motivations:

   a. "Stability" of holomorphic vector bundles: Why it is good. Generalization of holo vb's are objects in "derived category." Stability in the derived category is not known.

   b. (generalized) Donaldson-Thomas invariants: Enumerative algebraic geometry of Calabi-Yau 3-folds.

4. Wall-crossing: From BPS bound derive the notion of a wall of marginal stability.

   a. Remind people of the spectrum of BPS states in $SU(2), \mathcal{N}_f = 0$ SW, just to convince them that it certainly happens.

   b. Statement of the primitive wall-crossing formula.

   c. Statement of the semi-primitive wall-crossing formula.

   d. Kontsevich-Soibelman formula will be stated later.

5. Supergravity tools:

   a. Attractor mechanism.

   b. Basic trichotomy: regular attractor point, zero of Z, boundary. What it means for the spectrum. Entropy function and discriminant in the sugra regime.

   c. Review multi-centered solutions.

   d. Example of D4 splitting to D6-D6bar.

   e. Denef's split attractor flow conjecture.

   f. Another example: $D6\overline{D0}$ system.

   g. (Comment on walls of anti-marginal stability ?)

6. Derivation of the primitive wall-crossing formula from 2-centered solution. Explain the spin in the electromagnetic field of two dyons and observe that the boundstate radius goes to infinity.

7. Halo states: Example of D0 halos around a D6. Show that there is a Fock space: explain the oscillators. Derive the McMahon function. Generalize this and derive the semiprimitive wall-crossing formula.

8. Kontsevich-Soibelman formula.

   a. Statement: group of symplectomorphisms on K-theory torus.

   b. Construction of special group elements and statment of KS formula.

   c. Recover primitive wc formula.

   d. Show how to recover semi-primitive formula (maybe too detailed?)

   e. Give examples of $SU(2)$ SW theory for $\mathcal{N}_f = 0, 1, 2, 3$.

7. Lecture 3: Examples and Applications of BPS Wallcrossing

1. Wall-crossing in quiver gauge theories.

   a. Define quiver gauge theory and quantum mechanics.

   b. State claim that when constituent Z's are lined up light open string modes are effectively described by a quiver quantum mechanical system. (Regard this as an interesting conjecture motivating the study of these quiver systems.)

   c. Higgs branch: Kahler quotient and D-terms. FI terms and level of the moment map.

   d. Focus on the example of $U(1)$ QED coupled to $n_{\pm}$ chiral multiplets of charge ± 1.

   e. Explain that there is no classical coulomb branch, but that quantum effects lead to BPS states. Demonstrate this in the Hamiltonian picture by writing out the superchanges and using the Born-Oppenheimer approximation to derive the effective supersymmetry charges: These give the $|n_+ - n_-|$ BPS states on the Coulomb branch.

Show how they disappear from the spectrum when ϑ goes through zero. Recovering the picture in Denef's QQHHH paper.

f. Explain the spin paradox and resolution?

g. Other quiver examples ?

2. Applications to moduli of vector bundles. Explain paper with Diaconescu:

   a. Define a "stable sheaf." Example of an extension of rank one torsion free by rank one torsion free (D4 bound to D0).

   b. Now examine the relation between "physical stability" and "mathematical stability." Derive the walls for $D4$ to split to $D4 + D4$. Note that they only asymptote to the wall of mathematical stability:

   c. Recover the Gottsche-Yoshioka formulae.

   d. Main lesson: Even for a rigid surface, the moduli of D4's wrapping the surface is NOT the moduli space of slope-stable sheaves.

3. D6-D2-D0 system.

   a.) Gopakumar-Vafa invariants count D2D0 states.

   b.) Halos of D2D0 around D6.

   c.) The wall-crossing formula.

   d.) Resembles the Donaldson-Thomas partition function, which, naively counts all D6-D2-D0 states.

   e.) Explain that only for a specific B-field do you get the Donaldson-Thomas partition function.

   f.) Physical interpretation of the spin 0 vs. spin > 0 GV partition function in terms of core states and halo states.

   g.) New results with Aganagic-Jafferis (?)

8. The OSV conjecture

1. D4-D2-D0 system. P in Kahler cone. Dual description: M5 wrapping holomorphic surface. We are interested in $\Omega(\Gamma; t)$ - but for which t?

   ****

   Singleton decomposition with Narain lattice $L = \iota^*(H^2(X, \mathbb{Z}))$. Get this by holographic dual computation of M5 brane partition function via wavefunction of the M-theory C-field. Interpret sectors as Page charges.

   Remarks on polar states: Our understanding even of the polar region and certainly the polar degeneracies is much more rudimentary. Most important: The degeneracies *depend on the Kahler structure!*. This is the topic of the next chapters.

   ****

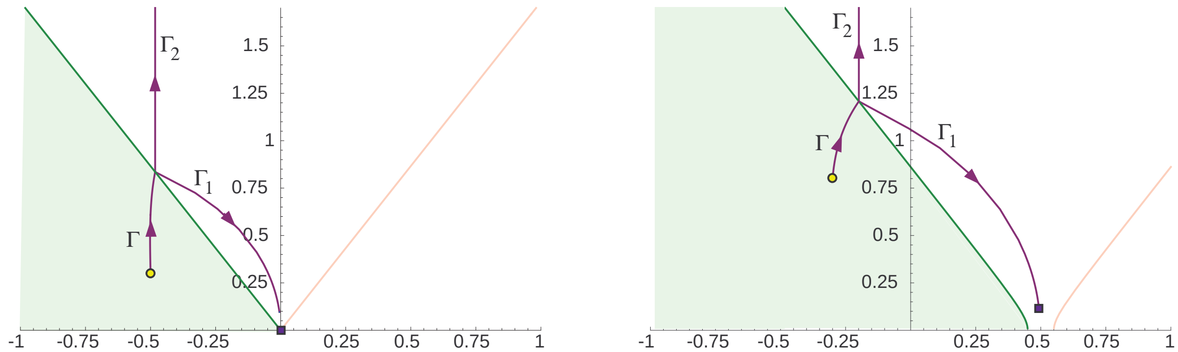

Figure 7: A standard choice of fundamental domain

2. $t = t_*(\Gamma)$ is the most relevant for the black hole entropy question. But we cannot say much because we cannot compute it microscopically. Microscopic computation of Ω available for $J \to \infty$.

3. Show that even

$$\lim_{J \to \infty} \Omega(\Gamma; B + iJ) \tag{8.1}$$

is not well-defined and one must specify a direction. Changes in direction can produce big change in entropy (Andryiash example.) Preferred direction $J = \lambda P$.

4. State OSV formula - rough version.

5. Microscopic description of the D4D2D0 system: Partition function. This is also the (0,4) elliptic genus.

6. Separate out the Narain partition function: Vector of modular forms of negative weight. $\Rightarrow$ apply Fareytail $\Rightarrow$ define "polar states."

7. Return to D4 splitting to D6-D6bar. Extreme polar state conjecture.

8. Barely polar states: Explain the "entropy enigma." Stress the weak g_{top} vs. strong g_{top} regimes. Mention relation to quiver picture: Head of a pin.

9. Statement of a precise version of an OSV formula.

10. Sketch the derivation. Emphasize the swing state conjecture and the core dump exponent as open problems.

11. Discussion of entropy enigma, $k = 2$ vs. $k = 3$, and the degeneracy dichotomy as an important open problem.

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