Framed BPS States In Four And Two Dimensions

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String-Math, Paris, June 27, 2016

1. **Review Derivation Of KS-WCF Using Framed BPS States**

   (with D. Gaiotto & A. Neitzke, 2010, …)

2. Interfaces in 2d N=2 LG models & Categorical CV-WCF

   (with D. Gaiotto & E. Witten, 2015)

3. Application to knot homology

   (with D. Galakhov, 2016)

4. Semiclassical BPS States & Generalized Sen Conjecture

   (with D. van den Bleeken & A. Royston, 2015; D. Brennan, 2016)

5. Conclusion

Basic Notation For d=4 N=2

Coulomb branch (special Kähler) $u \in \mathcal{B}$

Local system of infrared charges:
(flavor & electromagnetic) $\Gamma \to \mathcal{B}$

DSZ pairing: $\langle \cdot, \cdot \rangle : \Gamma_u \to \mathbb{Z}$

$$X_{\gamma_1} X_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1 + \gamma_2} \quad \gamma_1, \gamma_2 \in \Gamma_u$$

N=2 central charge. $Z : \Gamma \to \mathbb{C}$
Linear on Γ_u

$Z_\gamma(u)$

Supersymmetric Line Defects

Our line defects will be at $\mathbb{R}_t \times \{ 0 \} \subset \mathbb{R}^{1,3}$

A supersymmetric line defect L requires a choice of phase ζ :

Example: $L = \mathrm{Tr}_R \mathrm{Pexp} \int_{\mathbb{R}_t \times \vec{0}} (\zeta^{-1}\varphi + A + \zeta \bar{\varphi})$

$$\mathcal{H}_L = \oplus_{\gamma \in \Gamma + \gamma_0} \mathcal{H}_{L,\gamma}$$

Physical picture for charge sector γ:

An infinitely heavy BPS particle of charge γ at x=0.

Framed BPS States

$$E \geq -\mathrm{Re}(Z_\gamma/\zeta)$$

Framed BPS states are states in $\mathcal{H}_{L,\gamma}$ which saturate the bound.

$$\underline{\overline{\mathcal{H}}}^{\mathrm{BPS}}(\gamma; L, u) \subset \mathcal{H}_{L,\gamma}$$

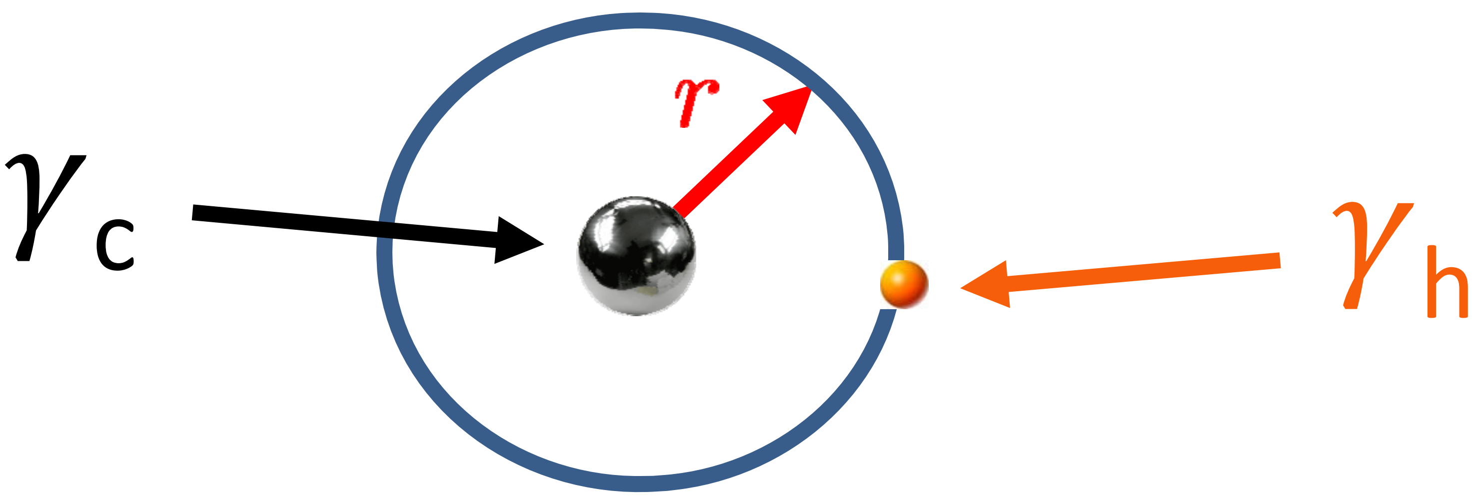
$$\underline{\overline{\Omega}}(\gamma; L, u) := \mathrm{Tr}_{\underline{\overline{\mathcal{H}}}^{\mathrm{BPS}}(\gamma;L,u)}(-1)^{2J_3}$$

So, there are *two* kinds of BPS states:

Ordinary/vanilla: $\Omega(\gamma; u)$

Framed: $\overline{\underline{\Omega}}(\gamma; L, u)$

Vanilla BPS particles of IR charge γ_h can bind to framed BPS states in IR charge sector γ_c to make new framed BPS states of IR charge $\gamma_c + \gamma_h$:



Framed BPS Wall-Crossing 1/2

Particles of charge γ_h bind to a ``core'' of charge γ_c at radius:

$$r = \frac{\langle \gamma_h, \gamma_c \rangle}{2\mathrm{Im}(Z_{\gamma_h}(u)/\zeta)}$$

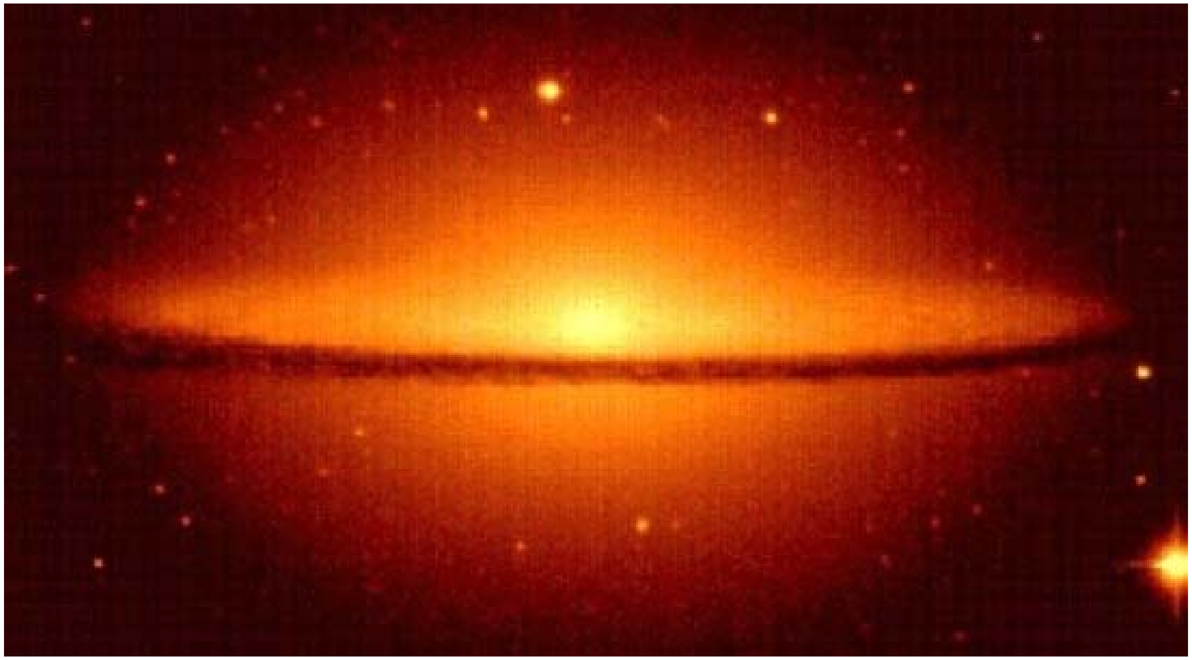
Define a <u>***``K-wall''***</u> :

$$W_{\gamma_h} := \{ \, u | \quad Z_{\gamma_h}(u) \parallel \zeta \quad \}$$

Crossing a K-wall the bound state comes (or goes).

Halo Fock Spaces

But, particles of charge γ_h, and also n γ_h for any n>0, can bind in arbitrary numbers: they feel no relative force, and hence there is an entire *Fock space* of boundstates with halo particles of charges n γ_h.



F. Denef, 2002

Denef & Moore, 2007

Framed BPS Wall-Crossing 2/2

So across the K-walls

$$W_{\gamma_h} := \{u| \quad Z_{\gamma_h}(u) \parallel \zeta \quad \}$$

<u>entire Fock spaces</u> of boundstates come/go.

Introduce ``Fock space creation/annihilaton operators'' for the Fock space of all bound vanilla BPS particles of charge n γ_h , n> 0 :

$$\mathfrak{K}(\gamma_h)$$

They operate on Hilbert spaces of framed BPS states

$$\mathfrak{K}(\gamma_h) : \overline{\underline{\mathcal{H}}}_{\gamma_c}^{\mathrm{BPS}} \to \overline{\underline{\mathcal{H}}}_{\gamma_c}^{\mathrm{BPS}} \widehat{\otimes} \mathcal{F}(\gamma_h)$$

"Annihilation": Near the K-wall the Hilbert space must factorize and

$$\mathfrak{K}(\gamma_h) : \overline{\underline{\mathcal{H}}}_{\gamma_c}^{\mathrm{BPS}} \widehat{\otimes} \mathcal{F}(\gamma_h) \to \overline{\underline{\mathcal{H}}}_{\gamma_c}^{\mathrm{BPS}}$$

Computing partition functions:

$$\mathfrak{K}(\gamma_h) \quad \Longrightarrow \quad K_{\gamma_h}^{\Omega(\gamma_h)}$$

$$K_{\gamma_h}(X_{\gamma_c}) = (1 - X_{\gamma_h})^{\langle \gamma_h, \gamma_c \rangle} X_{\gamma_c}$$ y=-1

This picture leads to a physical interpretation & derivation of the Kontsevich-Soibelman wall-crossing formula.

Gaiotto, Moore, Neitzke; Andriyash, Denef, Jafferis, Moore (2010); Dimofte, Gukov & Soibelman (2009)

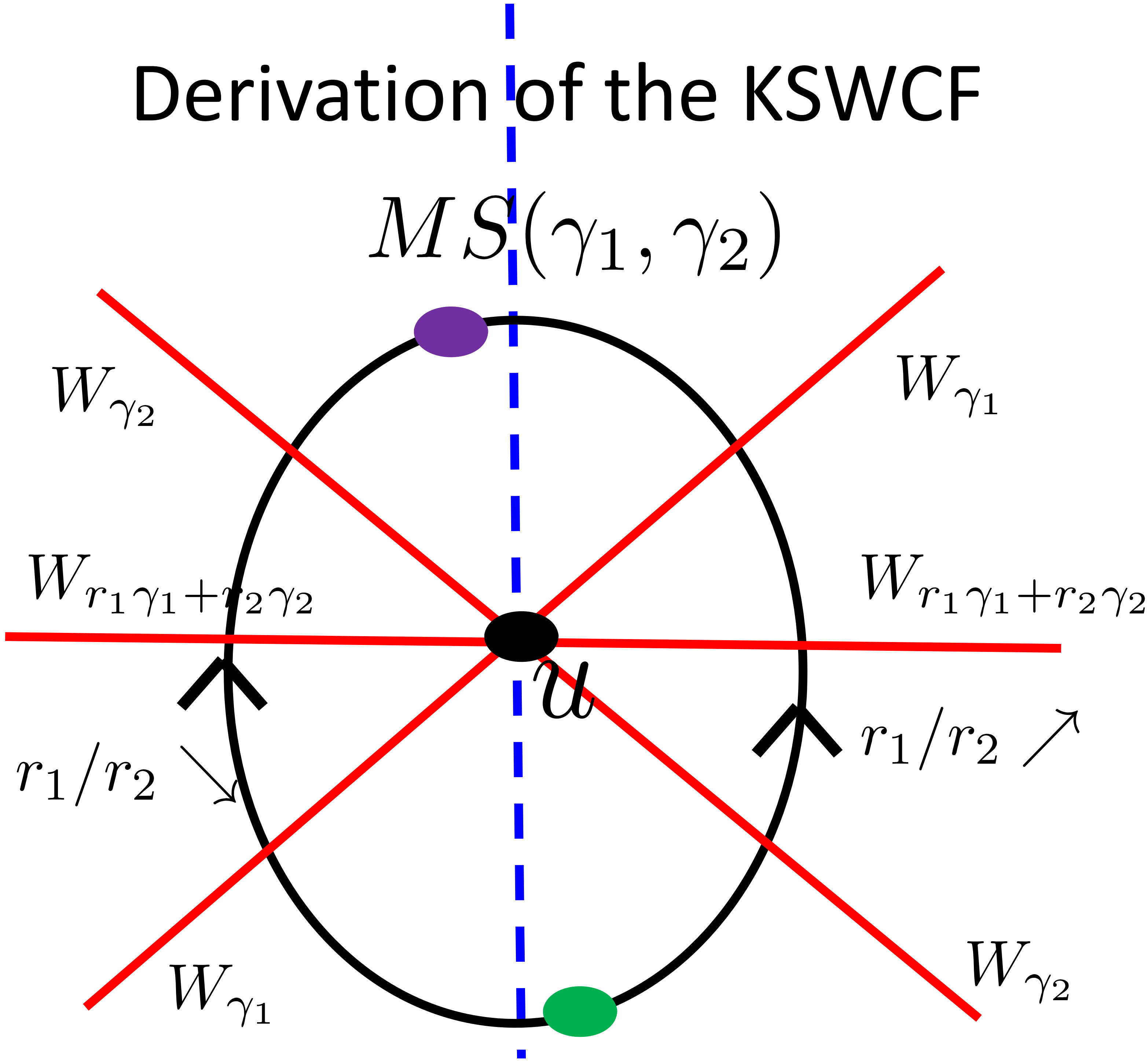
Consider a family of line defects along a path in $\mathcal{B}$

Suppose the path $\wp$ in the Coulomb branch $\mathcal{B}$ crosses walls $W_{\gamma_\alpha}, W_{\gamma_\beta}, \dots$

The BPS Hilbert space changes by the operation:

$$\cdots \mathfrak{K}(\gamma_\beta)\mathfrak{K}(\gamma_\alpha)$$

Derivation of the KSWCF



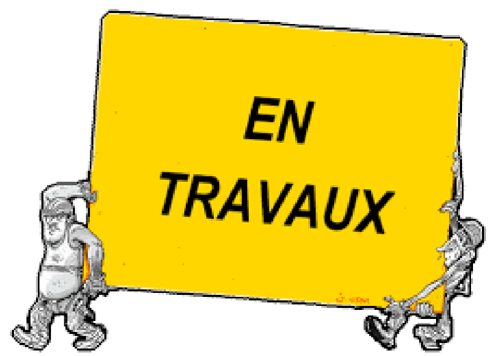
Categorified KS Formula ??

$$\mathfrak{K}^{-}(\gamma_2)\cdots\cdots\mathfrak{K}^{-}(\gamma_1)$$

$$\mathfrak{K}^{+}(\gamma_1)\cdots\cdots\mathfrak{K}^{+}(\gamma_2)$$

$\mathfrak{K}(\gamma_h) \to K_{\gamma_h}^{\Omega(\gamma_h)}$ gives the standard KSWCF.

Applied to BPS Hilbert space (considered as a complex with a differential) gives quasi-isomorphic spaces



Under discussion with T. Dimofte & D. Gaiotto.

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SQM & Morse Theory
(Witten: 1982)

M: Riemannian; h: *M* → $\mathbb{R}$, Morse function

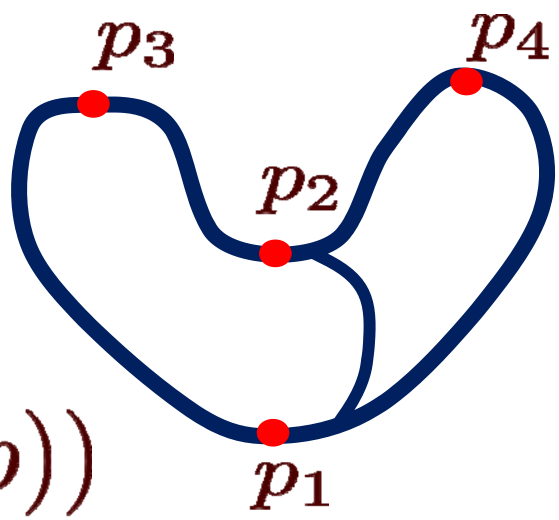
SQM: $q : \mathbb{R}_{\rm time} \to M \quad \chi \in \Gamma(q^*(TM \otimes \mathbb{C}))$

$$L = g_{IJ}\dot{q}^I \dot{q}^J - g^{IJ}\partial_I h \partial_J h$$

$$+g_{IJ}\bar{\chi}^I D_t \chi^J - g^{IJ} D_I D_J h \bar{\chi}^I \chi^J \; -R_{IJKL}\bar{\chi}^I \chi^J \bar{\chi}^K \chi^L$$

Perturbative vacua: $h'(p) = 0$

$\Longrightarrow \Psi(p)$



$$F(\Psi(p)) = \frac{1}{2}(d_{\downarrow}(p) - d_{\uparrow}(p))$$

Instantons & MSW Complex

Instanton equation: $$\frac{d\phi}{d\tau} = g^{IJ}\frac{\partial h}{\partial \phi^J}$$

Instantons lift some vacuum degeneracy.

To compute exact vacua:

MSW complex: $$\mathbb{M}^\bullet := \oplus_{p:h'(p)=0}\mathbb{Z}\cdot\Psi(p)$$

$$d(\Psi(p)) := \sum_{p':F(p')-F(p)=1} n(p,p')\Psi(p')$$

Space of groundstates (BPS states) is the *cohomology*.

LG Models

(X, ω) Kähler manifold.

$W : X \to \mathbb{C}$ Superpotential (A holomorphic Morse function)

$$\phi : \mathbb{R}^2 \to X$$

$$S = \int d\phi * d\bar{\phi} - |\nabla W|^2 + \cdots$$

Massive vacua are Morse critical points:

$$W'(\phi_i) = 0 \qquad W''(\phi_i) \neq 0$$

1+1 LG Model as SQM

Target space for SQM:

$$M = \mathrm{Maps}(\phi : \mathbb{R} \to X)$$

$$h = \int_{\mathbb{R}} \left(\phi^* \lambda + \mathrm{Re}(\zeta^{-1} W) dx\right)$$

$$d\lambda = \omega$$

Recover the standard 1+1 LG

Manifest susy:

$$\mathcal{Q}_\zeta = Q_- - \zeta^{-1} \bar{Q}_+ \qquad \bar{\mathcal{Q}}_\zeta = \bar{Q}_- - \zeta Q_+$$

Fields Preserving ζ-SUSY

Stationary points: ζ-*soliton* equation:

$$\delta h = 0 \quad \Longleftrightarrow \quad \frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

Gradient flow: $$\frac{d\phi}{d\tau} = \frac{\delta h}{\delta \phi}$$

ζ-*instanton* equation:

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

MSW Complex Of (Vanilla) Solitons

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

$\longrightarrow x$

$\phi \cong \phi_i$ $\qquad\qquad$ $\phi \cong \phi_j$

Solutions to BVP only exist when

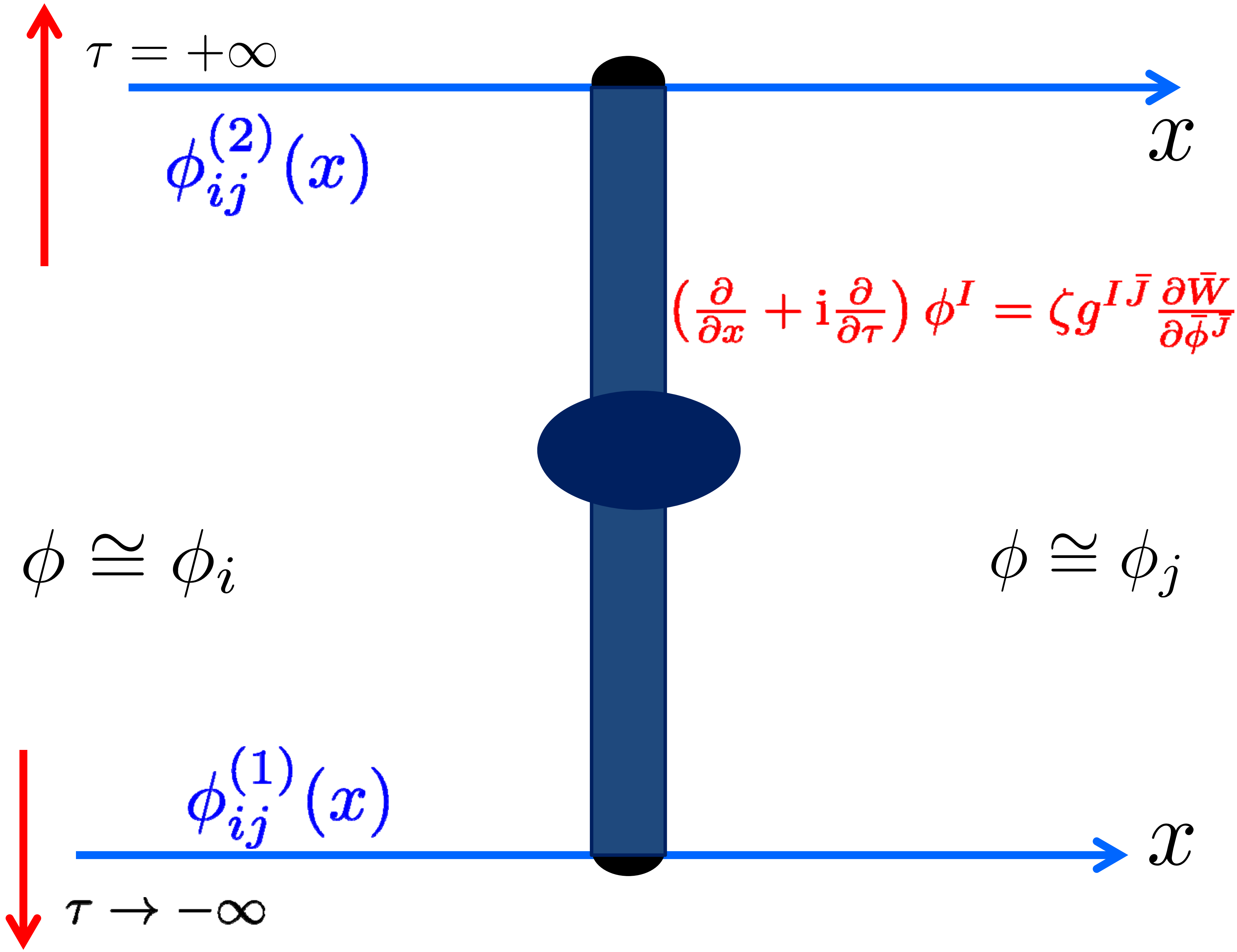
$$W_j - W_i \parallel \zeta \longleftarrow$$

♫ You must remember this

$$\mathbb{M}_{ij} := \oplus_{\text{solitons}} \mathbb{Z}\Psi[\phi_{ij}]$$

Matrix elements of the differential:

Count ζ -instantons


$\tau = +\infty$
$\phi_{ij}^{(2)}(x)$
x
$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi_{ij}^{(1)}(x)$
x
$\tau \to -\infty$

Families of Theories

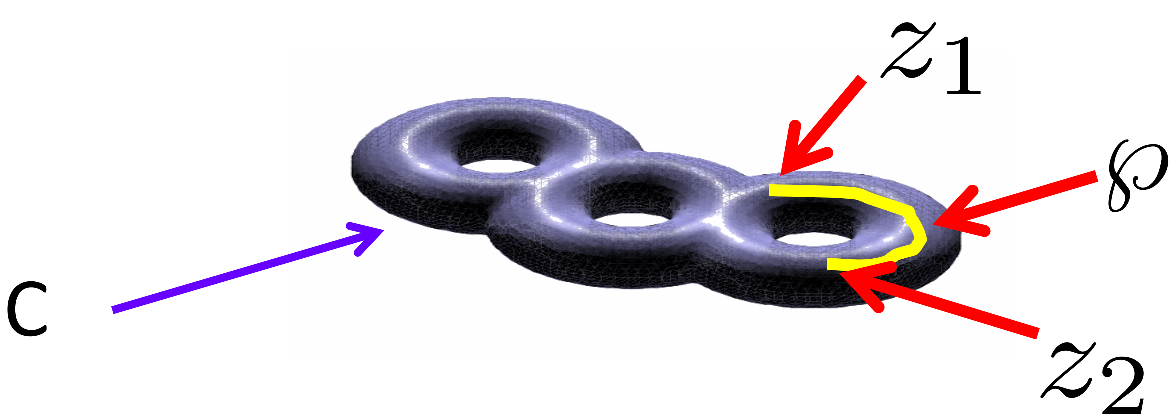
SQM viewpoint on LG makes construction of half-susy interfaces easy:

Consider a *family* of Morse functions

$$W(\phi; z) \quad z \in C$$

Let $\wp$ be a path in C connecting z_1 to z_2.

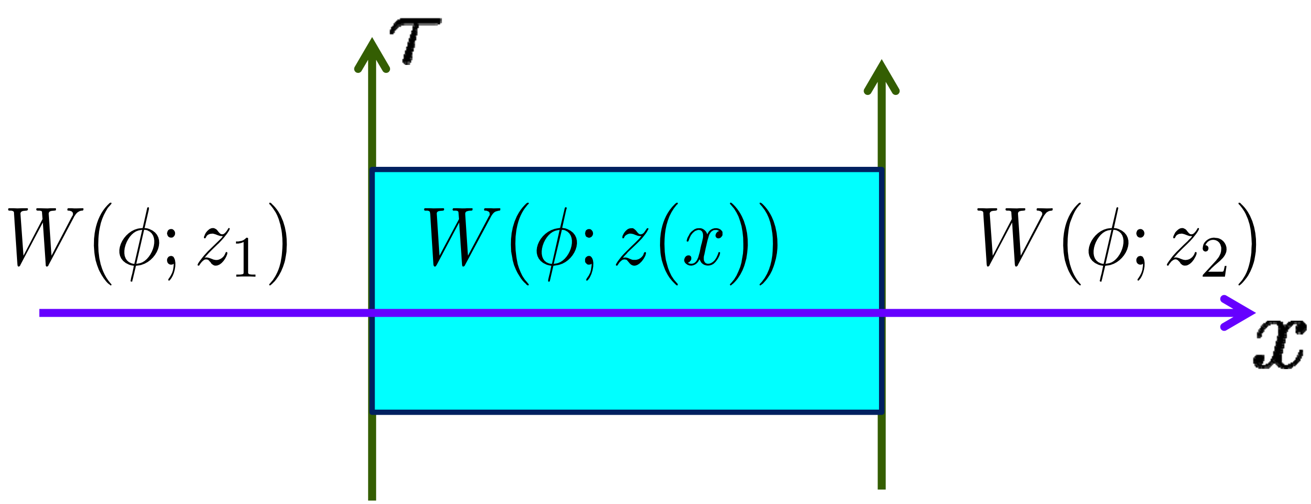
View it as a map $z: [x_l, x_r] \to C$ with $z(x_l) = z_1$ and $z(x_r) = z_2$



Domain Wall/Interface/Janus

Construct a 1+1 QFT (not translationally invariant) using:

$$h = \int_{\mathbb{R}} \phi^*(\lambda) + \mathrm{Re}(\zeta^{-1} W(\phi; z(x))) dx$$



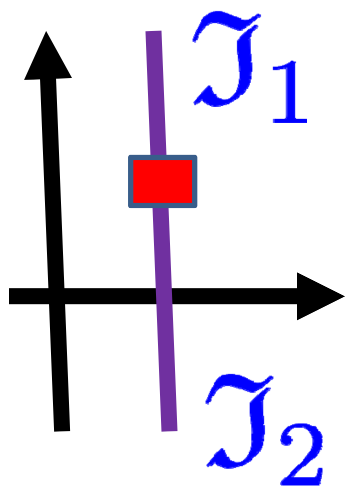
From this construction it manifestly preserves two supersymmetries.

General: A_∞-Category Of Interfaces

Interfaces between two theories (e.g. LG with different superpotentials) form an A_∞ category [GMW 2015]

$$\mathfrak{I} \in \mathfrak{Br}(\mathcal{T}_1, \mathcal{T}_2)$$

Morphisms between interfaces are local operators

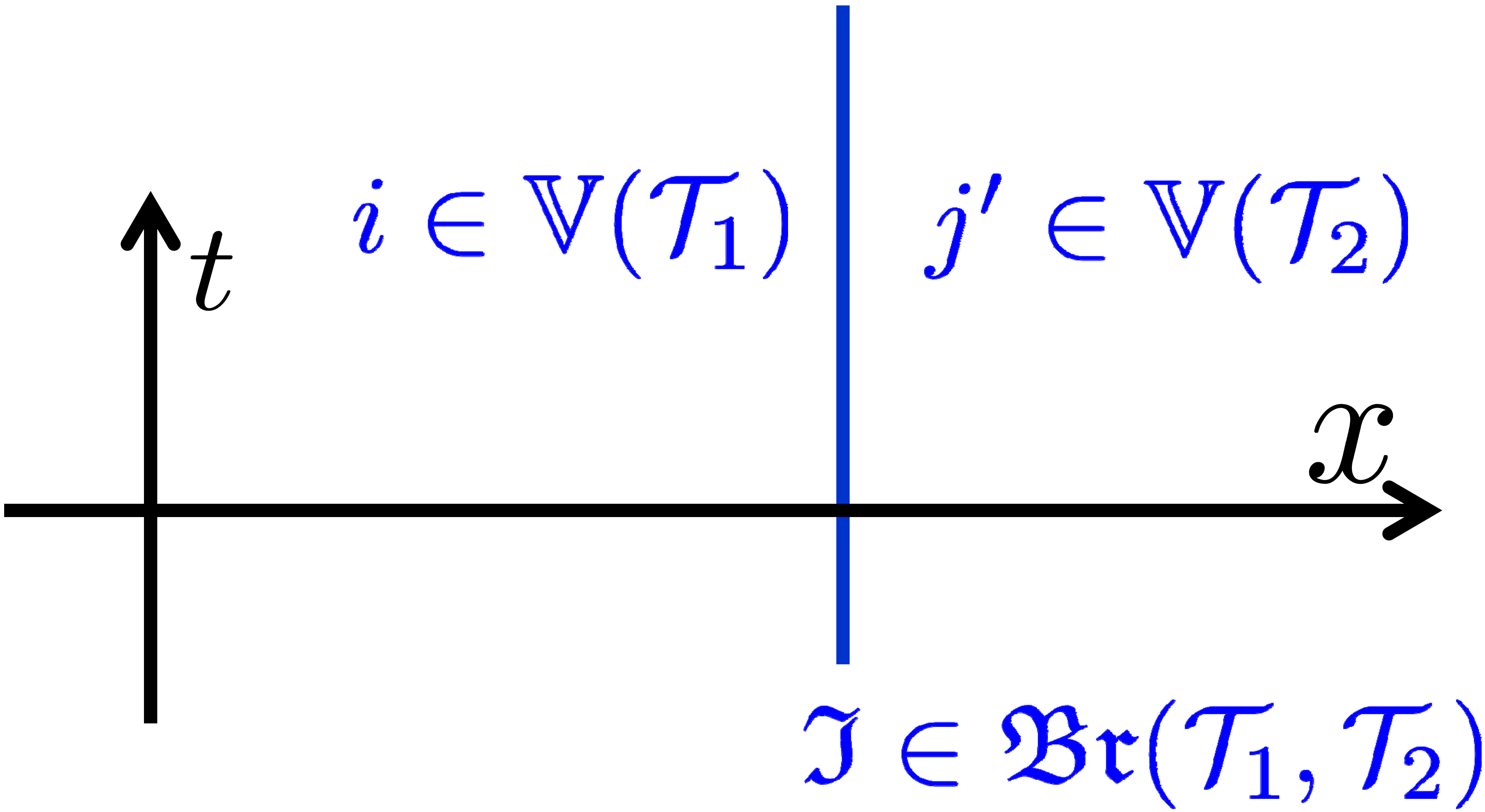


There is a notion of homotopy equivalence of interfaces $\mathfrak{I}_1 \sim \mathfrak{I}_2$

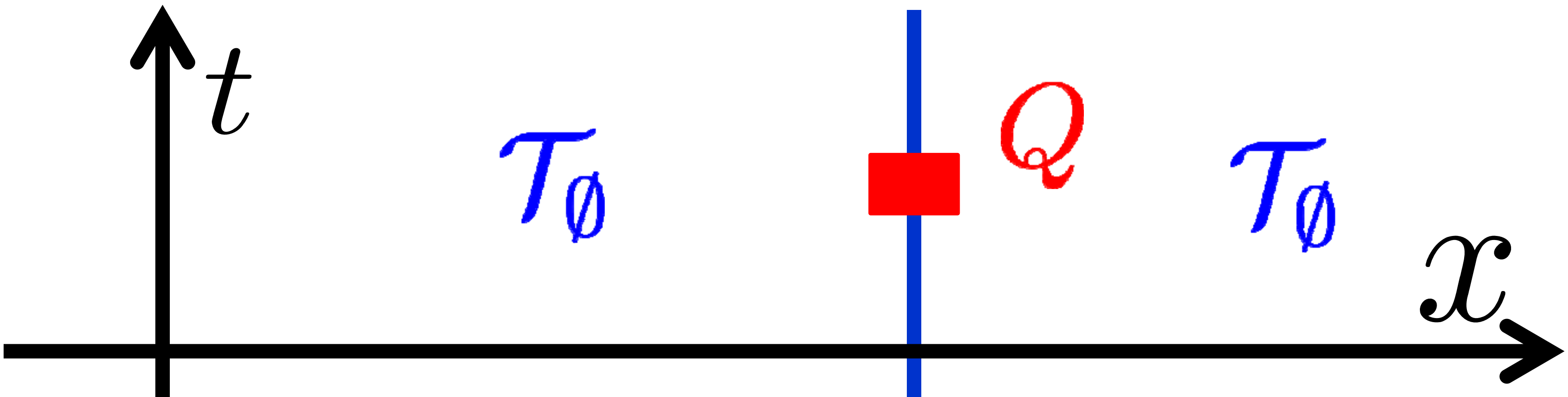
Means: There are boundary-condition changing operators invertible (under OPE) up to Q

Chan-Paton Data Of An Interface

$\mathcal{E}(\mathfrak{I})_{ij'}$ is a matrix of <u>complexes</u>.



Simplest Example



$Q : \mathcal{E} \to \mathcal{E}$

A_∞ equation: $Q^2 = 0$

$$\mathfrak{Br}(\mathcal{T}_\emptyset, \mathcal{T}_\emptyset) = \{\text{Chain complexes}\}$$

Interfaces For Paths Of LG Superpotentials

For LG interfaces defined by W(ϕ; z(x)) the matrix of CP complexes is the MSW complex of forced ζ-solitons:

$$\mathcal{E}(\mathfrak{I})_{ij'} = \oplus_{\textbf{forced solitons}} \mathbb{Z}\Psi_{ij'}$$

``Forced ζ – solitons'':

$$\frac{\partial}{\partial x}\phi = \zeta \overline{\frac{\partial W(\phi; z(x))}{\partial \phi}}$$

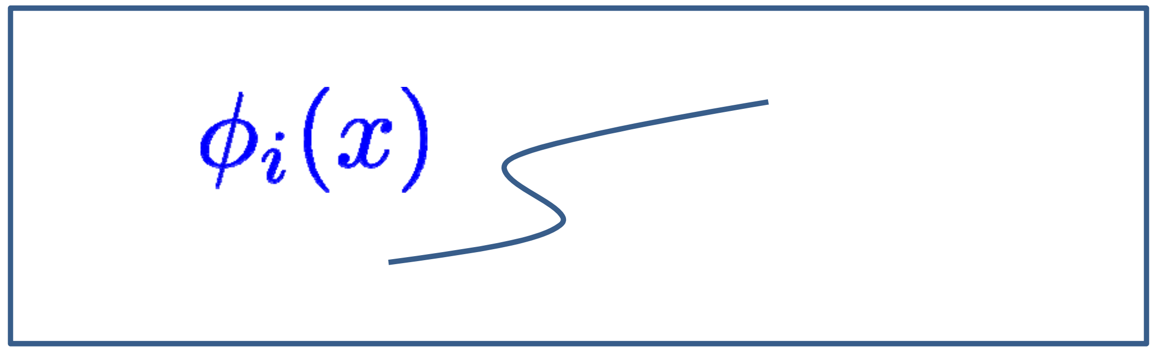
$$x \to -\infty \qquad\qquad x \to +\infty$$

$$W'(\phi_i; z_{\rm in}) = 0 \qquad\qquad W'(\phi_{j'}; z_{\rm out}) = 0$$

Hovering Solutions

For fixed x, the Morse function W(ϕ;z(x)) on X has critical points ϕ_i(x) that vary smoothly with x:

X



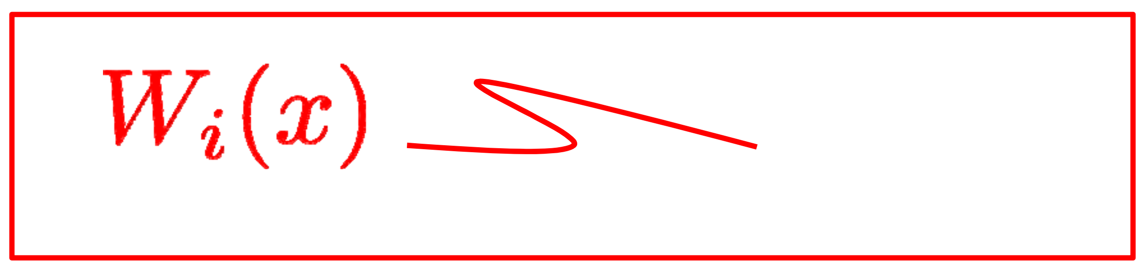
For adiabatic variation of parameters:

$$\left|\frac{dz}{dx}\right| \ll 1$$

$$\big\downarrow \; W(\phi; z(x))$$

these give the ``hovering solutions''

W-plane



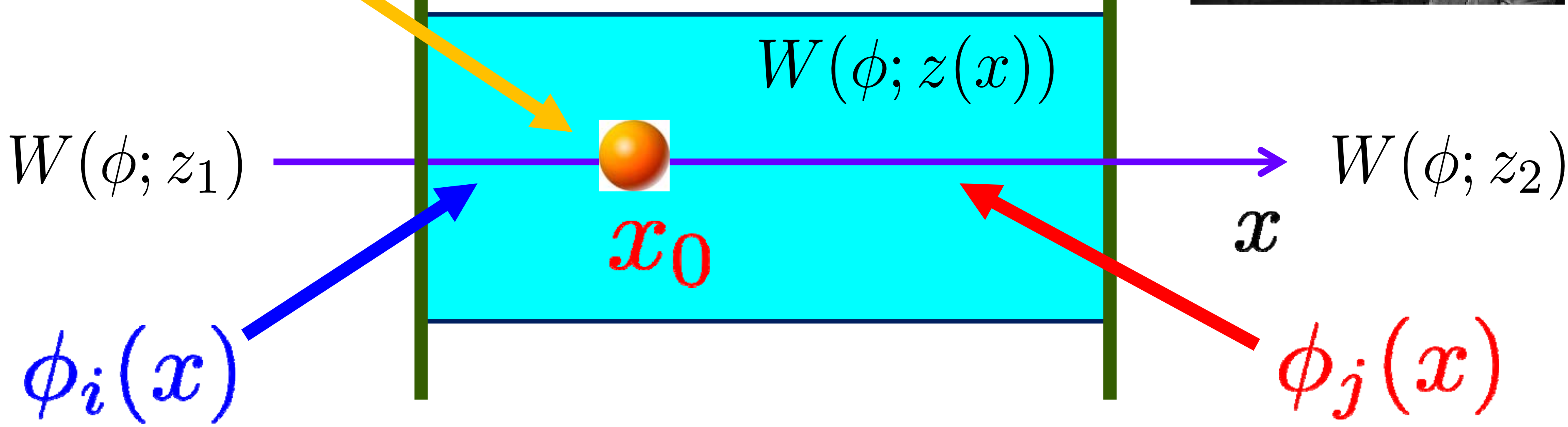
Binding Points

Critical values of W for theory @ z(x):

$$W_i(x) := W(\phi_i(x); z(x))$$

A <u>binding point</u> is a point x_0 so that:

$$W_j(x_0) - W_i(x_0) \parallel \zeta$$



S-Wall Interfaces

At a binding point a (*vanilla*!) soliton ϕ_{ij} has the *option* to bind to the interface, producing a new forced ζ-soliton:

These are the framed BPS states in two dimensions.

A small path crossing a binding point defines an interface

$$\mathfrak{S}_{ij}(x_0) \in \mathfrak{Br}(\mathcal{T}_{x_0-\epsilon}, \mathcal{T}_{x_0+\epsilon})$$

$$\mathcal{E}(\mathfrak{S}_{ij}(x_0)) = \mathbb{Z}\mathbf{1} + \mathbb{M}_{ij} e_{ij}$$

$\mathbb{M}_{ij}$ is the MSW complex for the (*vanilla*!) ζ-solitons in the theory with superpotential $W(\phi;z(x_0))$.

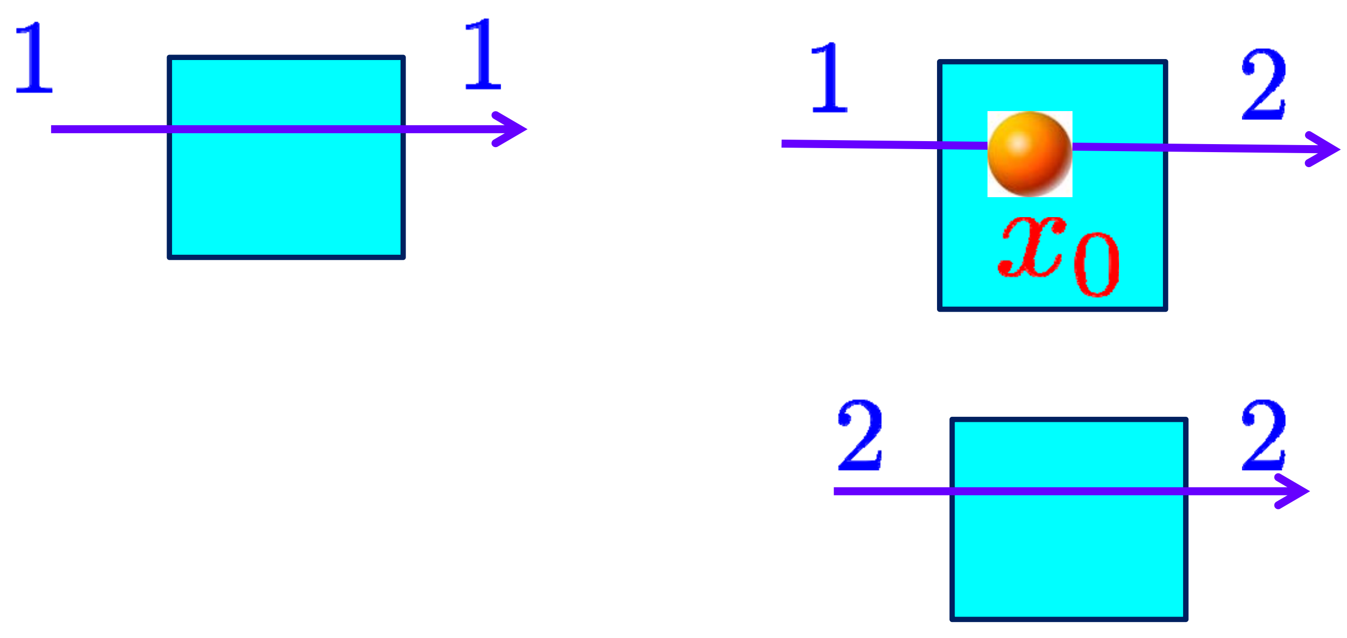
(In this way we categorify ``S-wall crossing'' and the ``detour rules'' of spectral network theory.)

Example Of S-Wall CP Data

Suppose there are just two vacua: 1,2

Suppose at the binding point x_0 there is one soliton of type 12, and none of type 21.

$$\mathcal{E}(\mathfrak{S}_{12}(x_0)) = \begin{pmatrix} \mathbb{Z}\cdot\Psi(1) & \mathbb{Z}\cdot\Psi(\phi_{12}) \\ 0 & \mathbb{Z}\cdot\Psi(2) \end{pmatrix}$$



Homotopy Property Of The Interfaces

For any continuous path $\wp$ of superpotentials:

$$W(\phi; z(x))$$

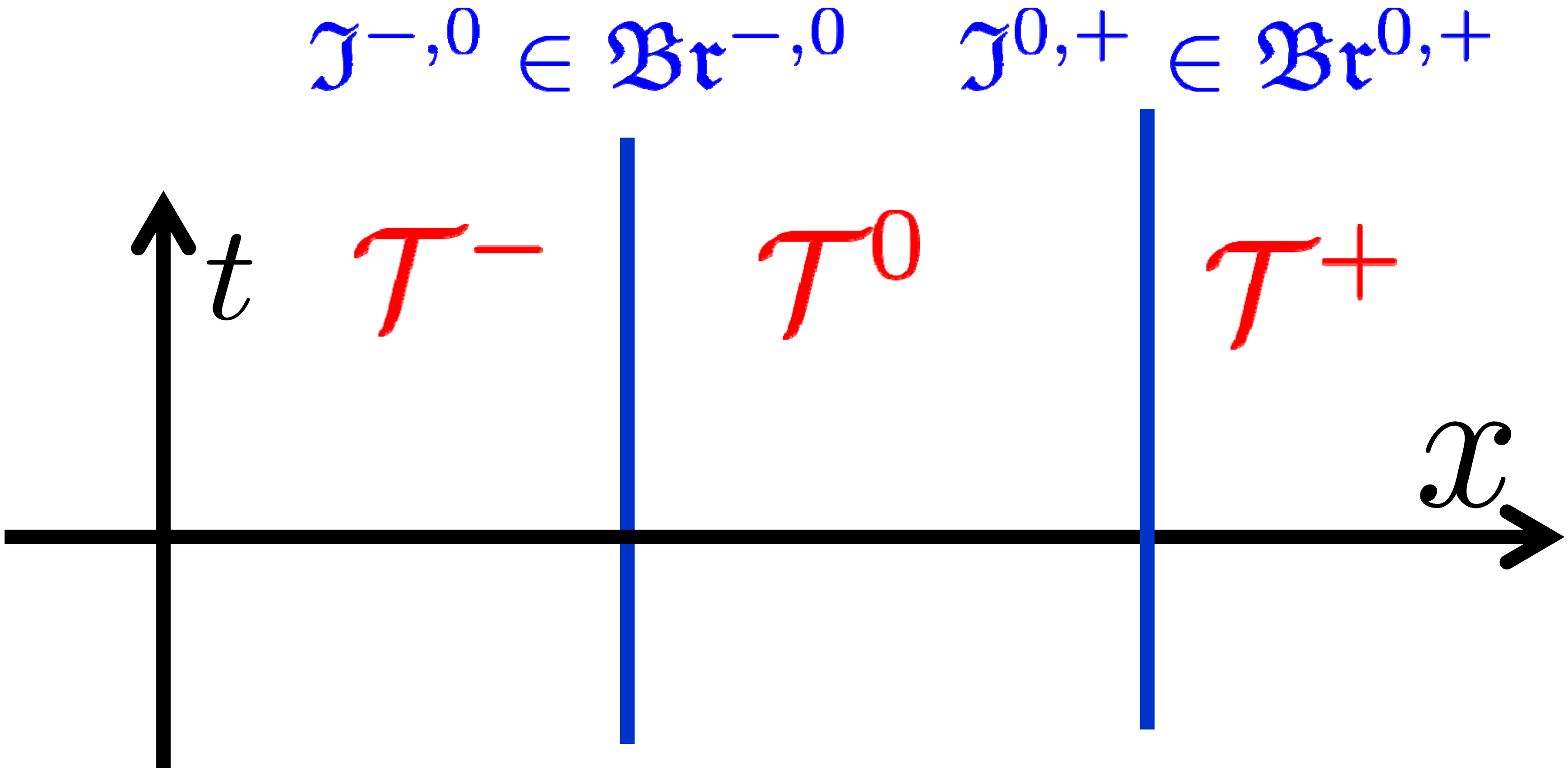
we have defined an interface:

$$\mathfrak{I}[\wp] \in \mathfrak{Br}(\mathcal{T}^{\text{in}}, \mathcal{T}^{\text{out}})$$

$$\wp \sim \wp' \quad \Rightarrow \quad \mathfrak{I}[\wp] \sim \mathfrak{I}[\wp']$$

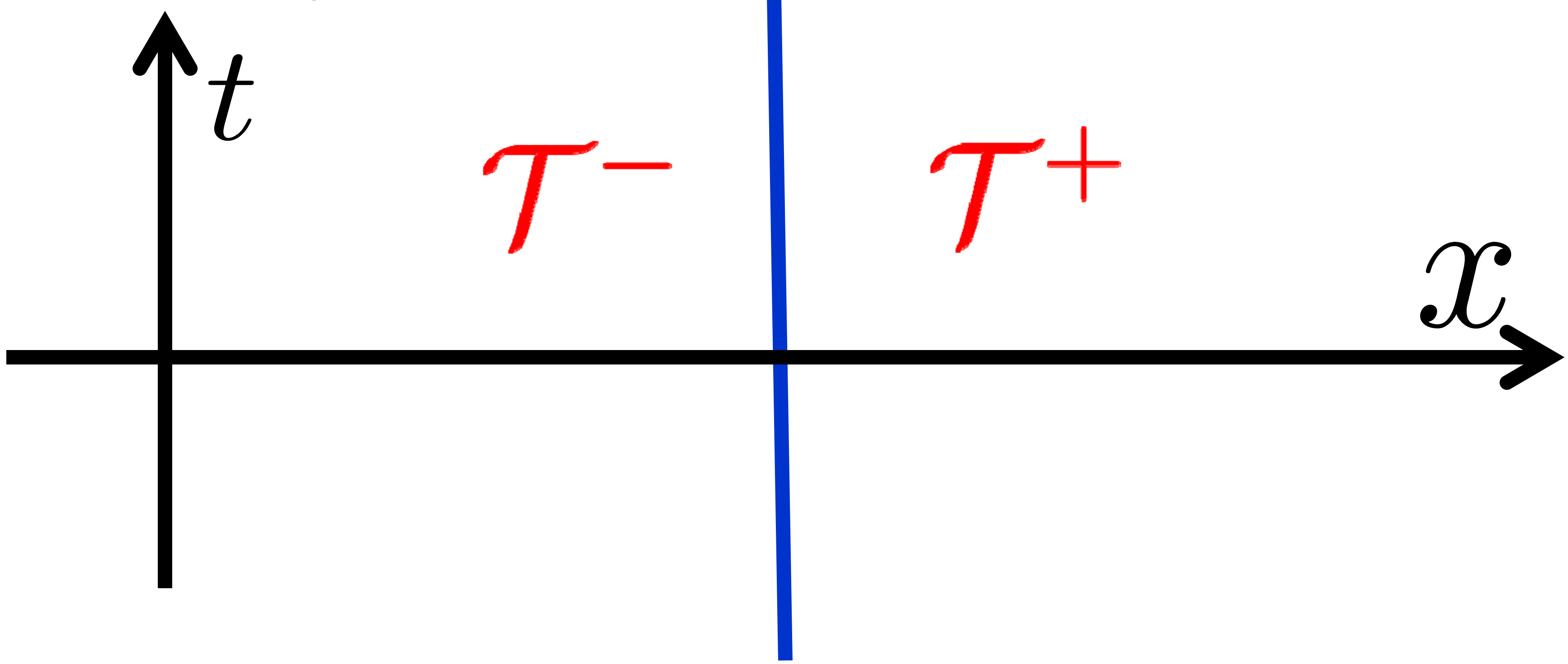
We want to use this to write the interface for $\wp$ in a simpler way:

Composition of Interfaces -1



GMW define a ``multiplication'' of the interfaces…

Composition of Interfaces - 2



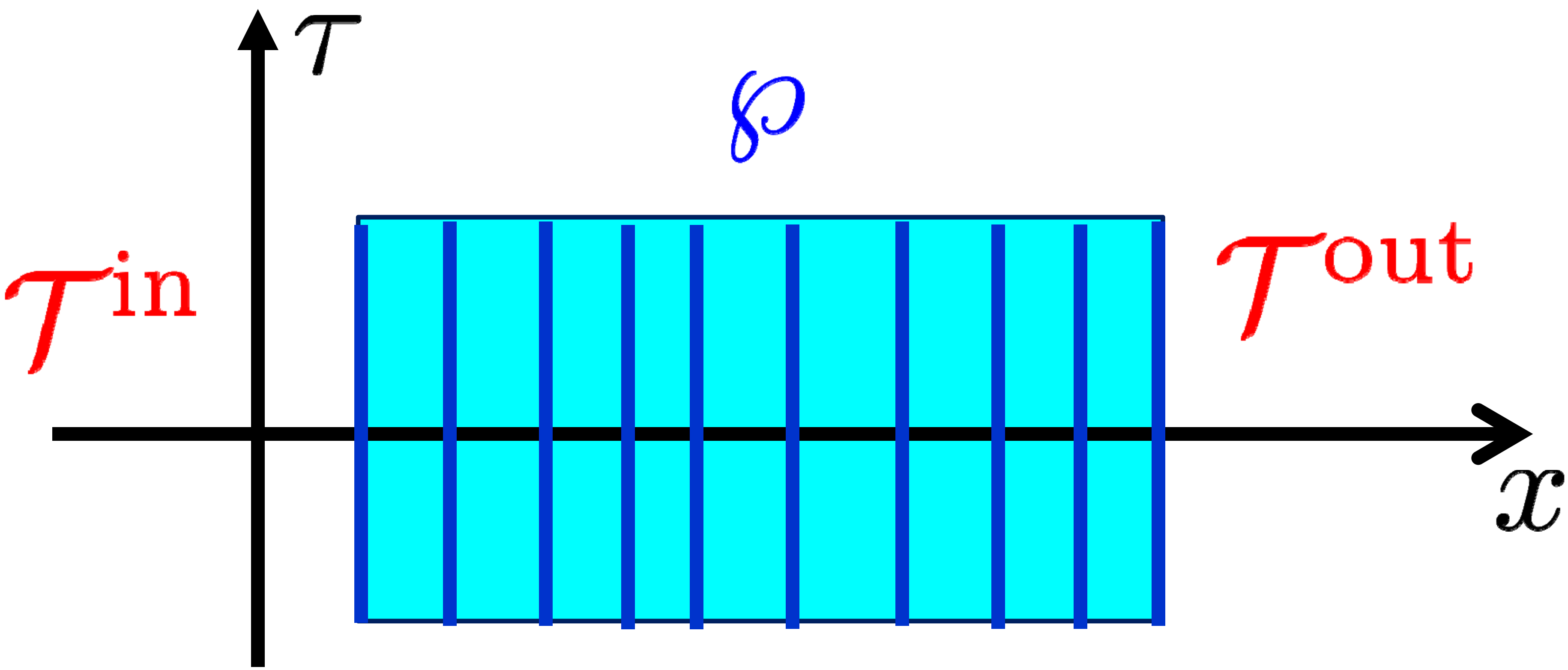
$$\mathfrak{I}^{-,+} = \mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+} \in \mathfrak{Br}^{-,+}$$

$$\mathcal{E}(\mathfrak{I}^{-,+}) = \mathcal{E}(\mathfrak{I}^{-,0})\mathcal{E}(\mathfrak{I}^{0,+})$$

But the differential is not the naïve one!

Reduction to Elementary Interfaces:

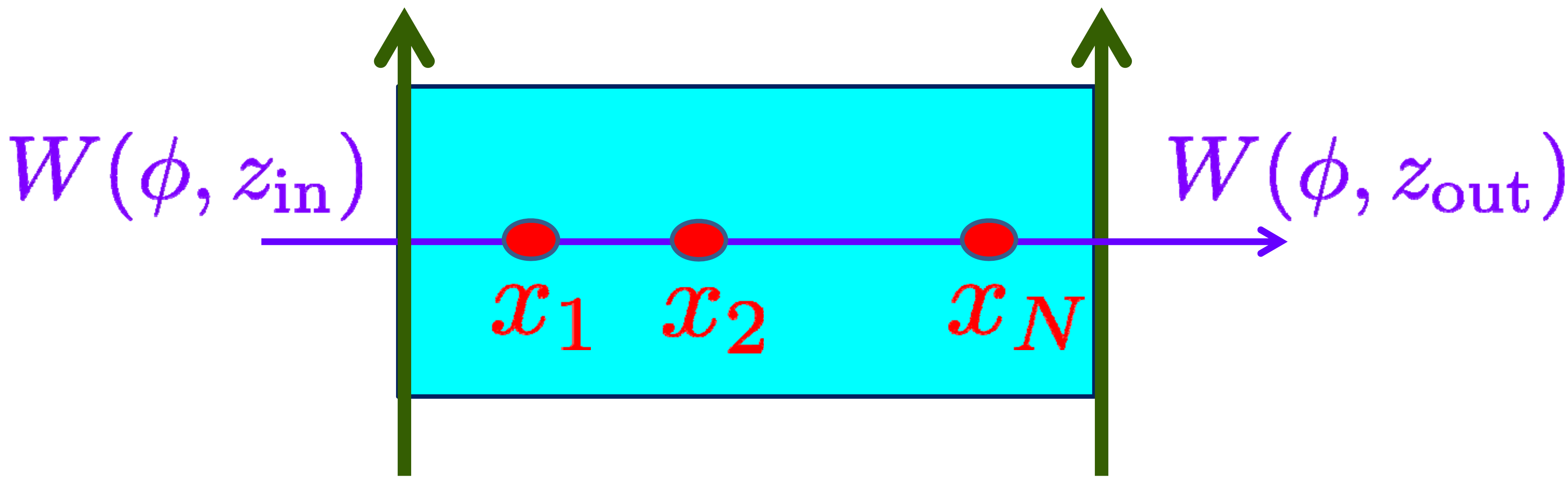
So we can now try to "factorize" the interface by factorizing the path:



$$\mathfrak{I}[\wp] \sim \mathfrak{I}_1 \boxtimes \cdots \boxtimes \mathfrak{I}_n$$

Factor Into S-Wall Interfaces

Suppose a path z(x) contains binding points: $x_1 < \cdots < x_N$



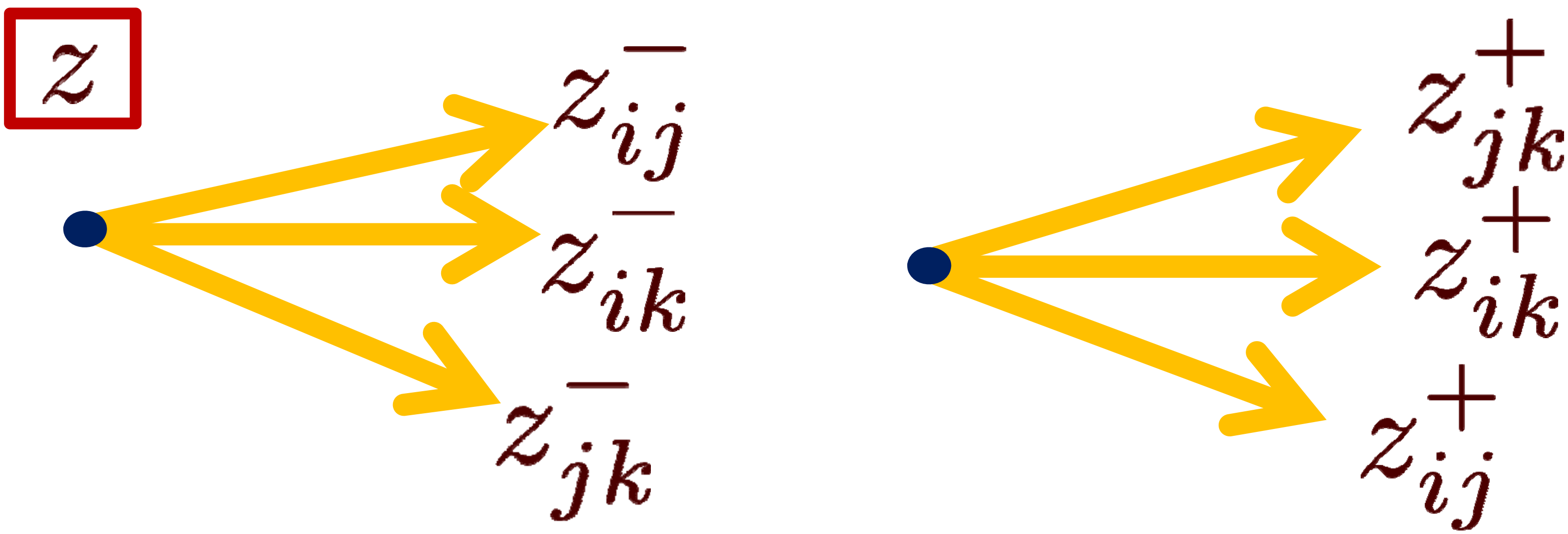
$$\mathfrak{I}[\wp] \sim \mathfrak{S}_{i_1 j_1}(x_1) \boxtimes \cdots \boxtimes \mathfrak{S}_{i_N j_N}(x_N)$$

(up to $\mathcal{A}_\infty$ equivalence of categories)

Categorified Cecotti-Vafa WCF -1/3

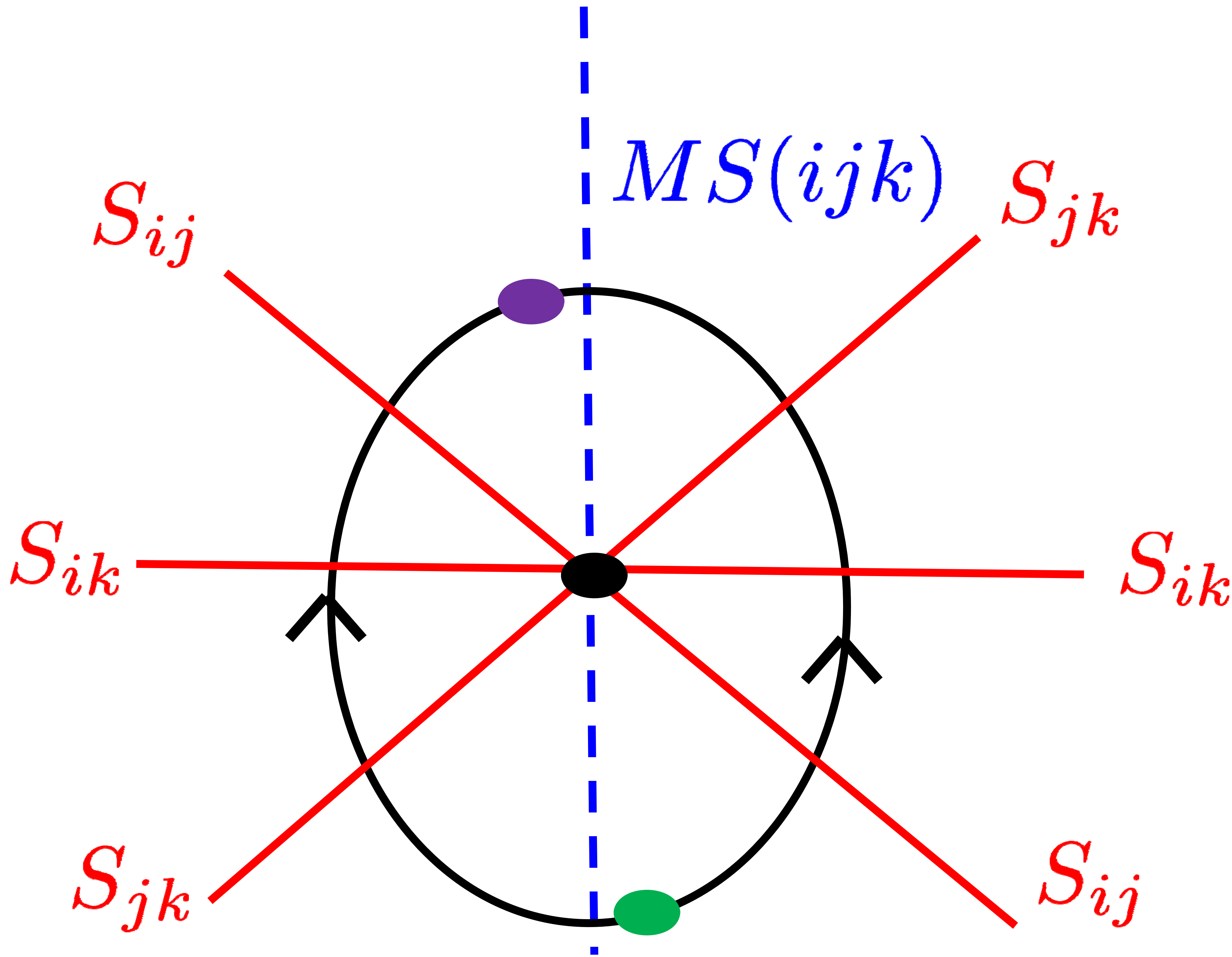
In a parameter space of superpotentials define walls:

$$MS(ijk) = \{param's | z_{ij} \parallel z_{jk}\}$$



Also define ``S-walls'' (analogs of ``K-walls'' in 4d) :

$$S_{ij}(\zeta) = \{param's | z_{ij} \parallel \zeta\}$$


$MS(ijk)$
S_{ij}
S_{jk}
S_{ik}
S_{ik}
S_{jk}
S_{ij}

Categorified Cecotti-Vafa WCF -3/3

$$\mathfrak{S}_{jk}^{-} \boxtimes \mathfrak{S}_{ik}^{-} \boxtimes \mathfrak{S}_{ij}^{-} \sim \mathfrak{S}_{ij}^{+} \boxtimes \mathfrak{S}_{ik}^{+} \boxtimes \mathfrak{S}_{jk}^{+}$$

(up to A_∞ equivalence of categories)

So, for the Chan-Paton data:

$$\mathcal{E}_{jk}^{-} \cdot \mathcal{E}_{ik}^{-} \cdot \mathcal{E}_{ij}^{-} \sim \mathcal{E}_{ij}^{+} \cdot \mathcal{E}_{ik}^{+} \cdot \mathcal{E}_{jk}^{+}$$

Up to quasi-isomorphism of chain complexes.

Witten index:

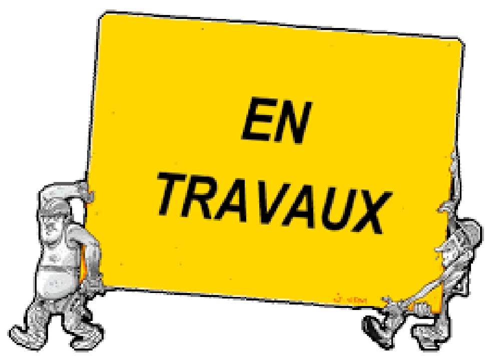
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{\mu_{jk}^{-}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\mu_{ik}^{-}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\mu_{ij}^{-}}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\mu_{ij}^{+}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\mu_{ik}^{+}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{\mu_{jk}^{+}}$$

A 2d4d Categorified WCF?

GMN 2011 wrote a hybrid wcf for BPS indices of both 2d solitons and 4d bps particles.

An ongoing project with Tudor Dimofte and Davide Gaiotto has been seeking to categorify it:



One possible approach: Reinterpret S-wall interfaces as special kinds of functors: They are mutation functors of a category with an exceptional collection.

We are seeking to define analogous ``K-wall functors'' .

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Knot Homology -1/3

(Approach of E. Witten, 2011)

Study (2,0) superconformal theory based on Lie algebra $\mathfrak{g}$

$$\mathbb{R} \times M_3 \times D$$

TIME x^0

CIGAR

D

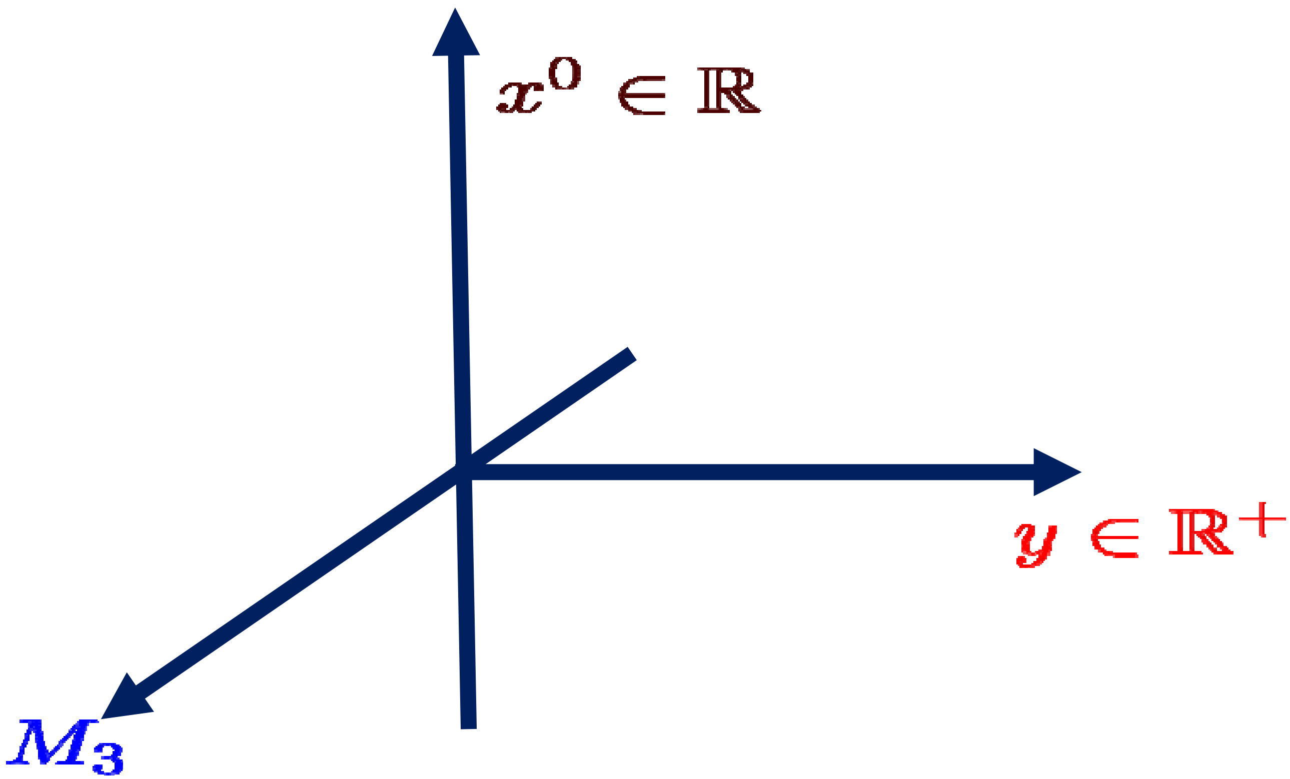
p

M_3: 3-manifold containing a link L

(Surface defect wraps $\mathbb{R}$ x L x {p})

Knot Homology – 2/3

Now, KK reduce by U(1) isometry of the cigar D with fixed point p to obtain 5D SYM on
$\mathbb{R} \times M_3 \times \mathbb{R}_+$



Knot Homology – 3/3

Hilbert space of states depends on M_3 and L:

$$\mathcal{H}_{\mathrm{BPS}}(M_3, L)$$

is the ``knot'' (better: link) homology of L in M_3.

This space is constructed from a chain complex using infinite-dimensional Morse theory:

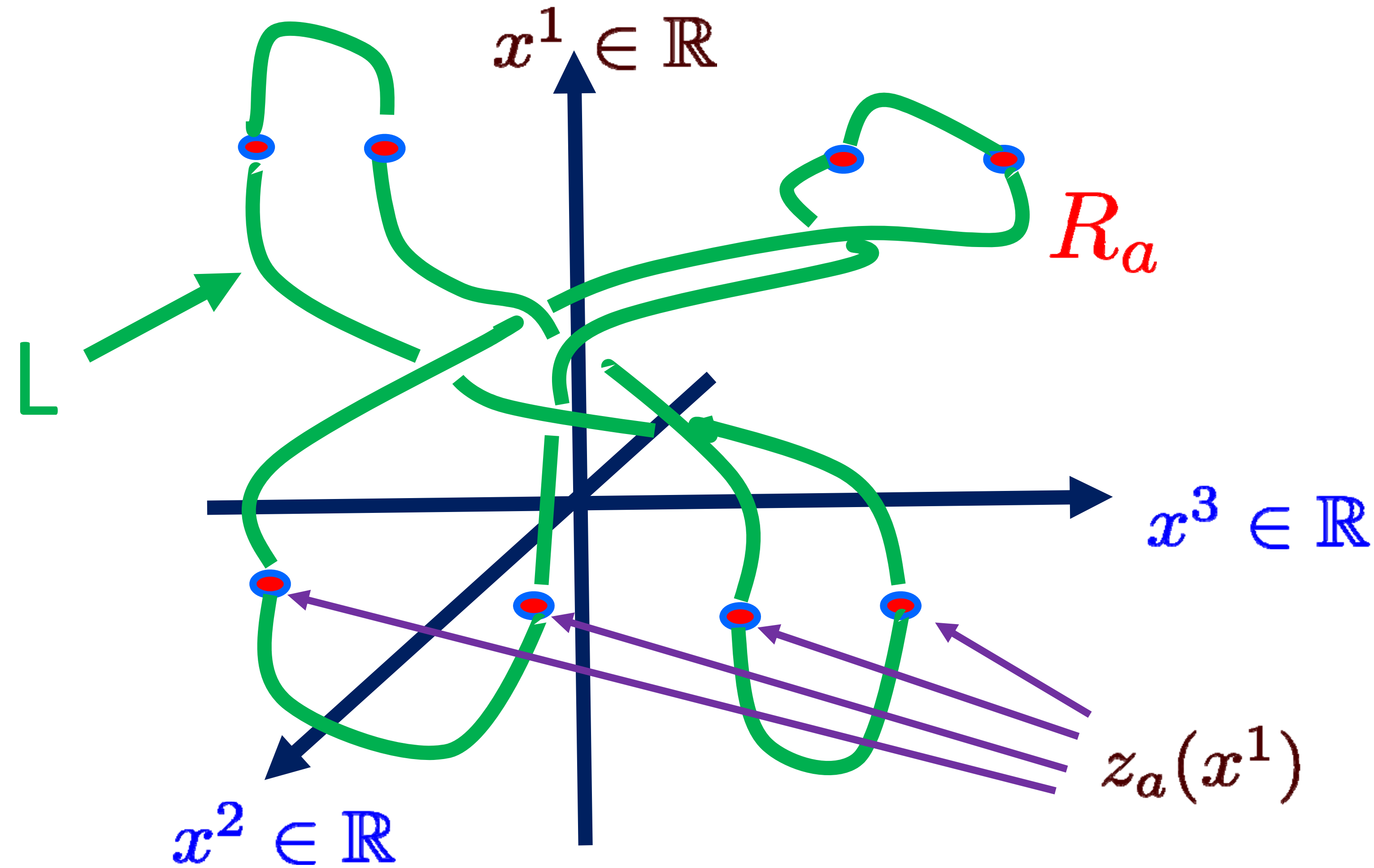
``Solitons'': Solutions to the Kapustin-Witten equations.

``Instantons'': Solutions to the Haydys-Witten equations.

Very difficult 4d/5d partial differential equations: Equivariant Morse theory on infinite-dimensional target space of (complexified) gauge fields.

Gaiotto-Witten Reduction

View the link as a *tangle*:
An evolution of complex numbers $z_a(x^1)$



Gaiotto-Witten Model 1: YYLG

Claim: When G=SU(2) and z_a do not depend on x^1 the Morse complex based on KW/HW equations is equivalent to the MSW complex of a finite dimensional LG theory in the (x^0,x^1) plane: ***YYLG model***:

$$(w_1,\dots,w_m) \in \bar{X} = \mathrm{Conf}_m(\mathbb{C} - \{z_a\})$$

w_i , i=1,…,m : Fields of the LG model

$$W = \sum_{i,a} k_a \log(w_i - z_a) - \sum_{i<j} \log(w_i - w_j)^2 + c\sum_i w_i$$

R_a = su(2) irrep of dimension k_a+1

z_a a= 1,… Parameters of the LG model

Variations of parameters: $z_a(x^1)$

give interfaces between theories

Gaiotto-Witten Model 2: Monopoles

$$X = \mathbb{R}^4 \times \mathcal{M}_0$$

Moduli space of smooth SU(2) monopoles on $\mathbb{R}^3$ of charge m

$$\bar{X} = \mathrm{RatMap}^m(\mathbb{CP}^1) \qquad = \{(P(u), Q(u))\}$$

$$Q(u) = \prod_{i=1}^m (u - w_i) \qquad K(u) = \prod_{i=1}^n (u - z_a)^{k_a}$$

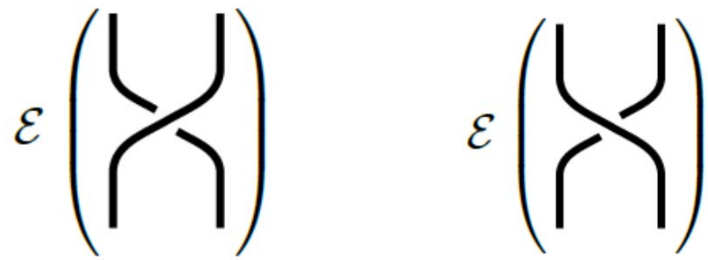
$$W = \sum_{i=1}^m \mathrm{Res}_{u=w_i} \frac{K(u)P(u)}{Q(u)} - \log P(w_i) + c w_i$$

Integrate out P:

$\Rightarrow$ Recover YYLG model.

Braiding & Fusing Interfaces

Braiding Interface:



Cup & Cap Interfaces: $\mathcal{E}\left(\bigcap\right)$ $\mathcal{E}\left(\bigcup\right)$

A tangle gives an x^1-ordered set of braidings, cups and caps.

Proposal for link chain complex

Let the corresponding x^1-ordered sequence of interfaces be

$$\mathfrak{I}_1, \ldots, \mathfrak{I}_N$$

$$\mathfrak{I}^{\mathrm{Link}} := \mathfrak{I}_1 \boxtimes \cdots \boxtimes \mathfrak{I}_N$$

is an Interface between a trivial theory and itself,

$$\mathfrak{I}^{\mathrm{Link}} \in \mathfrak{Br}(\mathcal{T}_\emptyset, \mathcal{T}_\emptyset) \quad = \{\text{Chain complexes}\}$$

So it is a ***chain complex***.

The Link Homology

The link (co-)homology is then:

$$H_L := H^{*,*}(\mathfrak{I}^{\mathrm{Link}}, Q^{\mathrm{Link}})$$

The link (co-)homology is bigraded:

$$P = -\frac{1}{\mathrm{i}\pi}\oint dW$$

F = Fermion number

Poincare polynomial:

$$P(q,t) = \mathrm{Tr}_{H_L} q^P t^F$$

(Chern-Simons) knot polynomial:

$$\chi(q) = \mathrm{Tr}_{H_L} q^P (-1)^F$$

Vacua For YYLG

Vacuum equations of YYLG

$$\sum_a \frac{k_a}{w_i - z_a} - 2\sum_{j\neq i} \frac{1}{w_i - w_j} + c = 0$$

Large c and k_a=1:

$$w_i = z_{a(i)} - \frac{1}{c} + \mathcal{O}(c^{-2})$$

Points z_a are unoccupied (-) or occupied (+) by a single w_i.

Example: Two z's & One w

+, - like spin up,down in two-dimensional rep of $SU(2)_q$

Recovering The Jones Polynomial

The relation to $SU(2)_q$ goes much deeper and a key result of the Gaiotto-Witten paper:

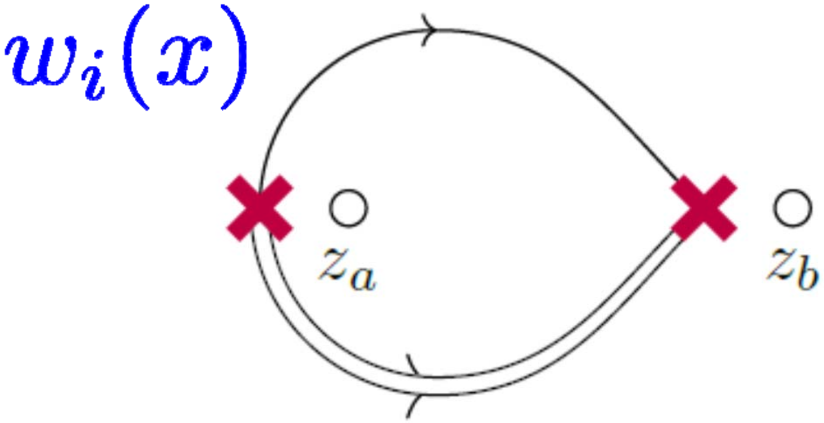
$$\chi(H^{*,*}(\mathfrak{J}^{\mathrm{Link}}, Q^{\mathrm{Link}})) = J_L(q)$$

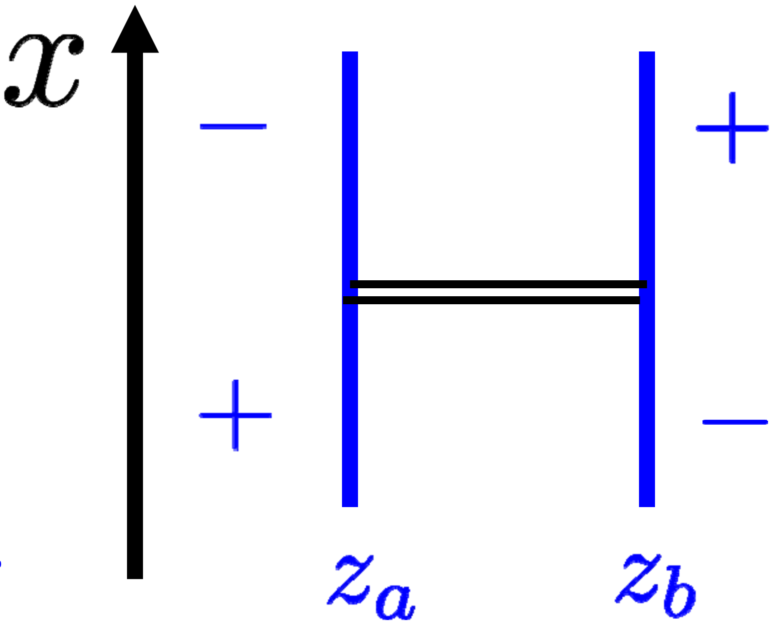
But the explicit construction of knot homologies in this framework remained open.

Computing Knot Homology

This program has been taken a step further in a project with Dima Galakhov.

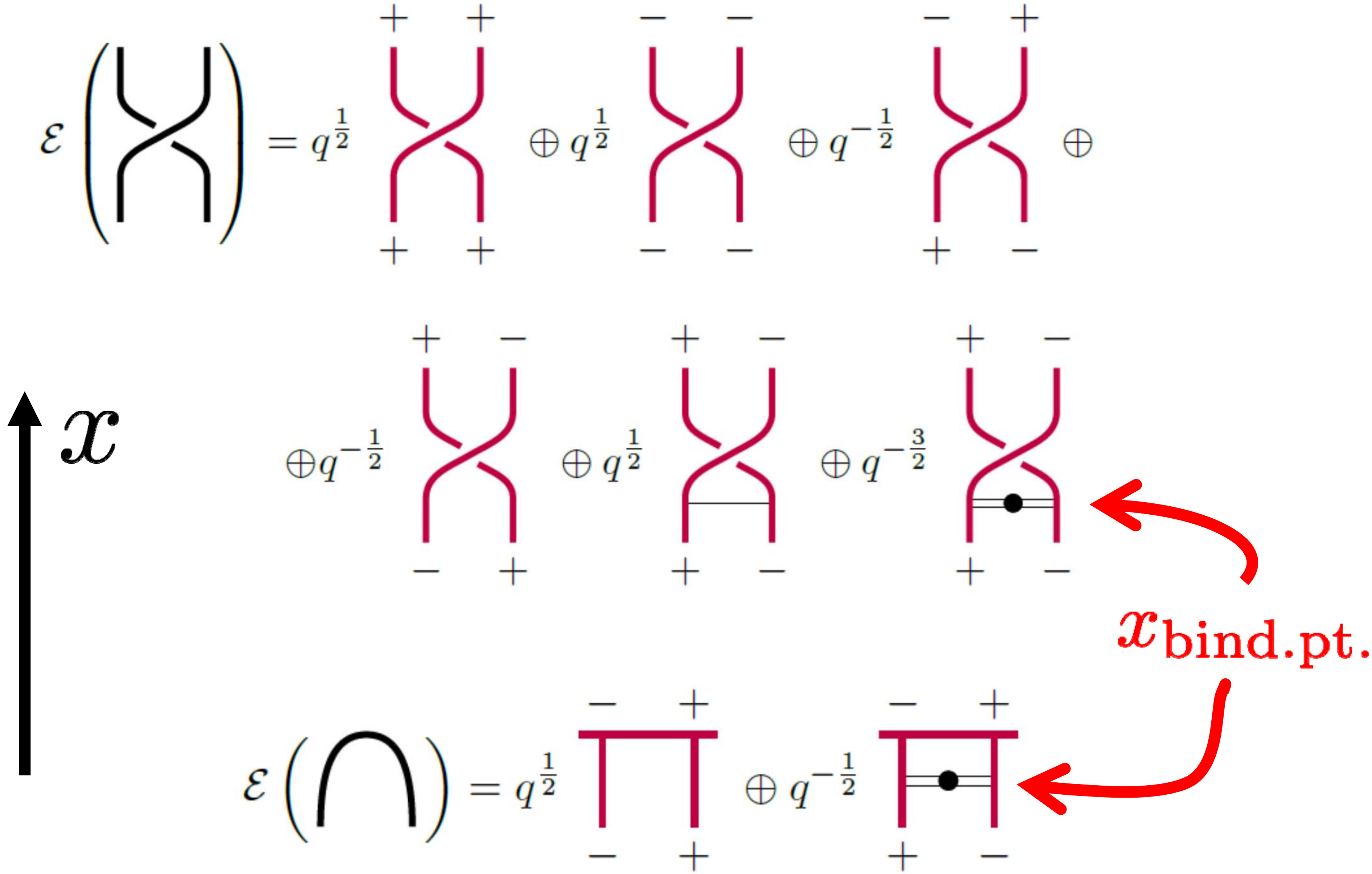
YYLG solitons: (…, + , - , ….) to (…, - , + , …)





All other $w_j(x)$ approximately constant.

Chan-Paton Data For Basic Moves



Bi-Grading Of Complex

The link homology complex is supposed to have a bi-grading.

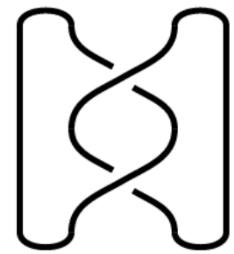
$$P = -\frac{1}{\mathrm{i}\pi}\oint dW \qquad F = \oint \omega$$

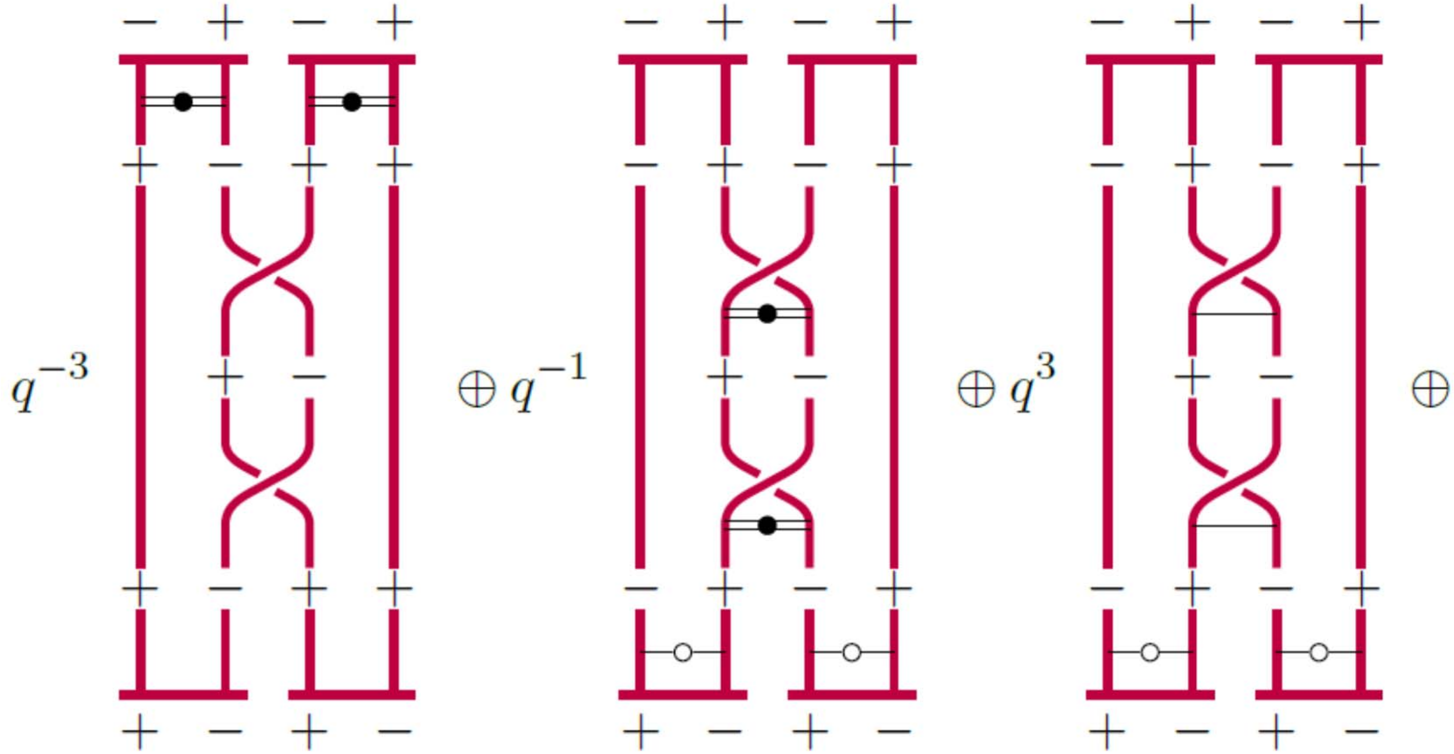
where ω is a one-form extracted from the asymptotics of the CFIV ``new'' supersymmetric index for interfaces:

$$Q_{ij'} = \mathrm{Tr}_{\mathcal{H}_{ij'}}(-1)^F e^{-\beta H[w_i(x);\zeta,z_a(x)]}$$

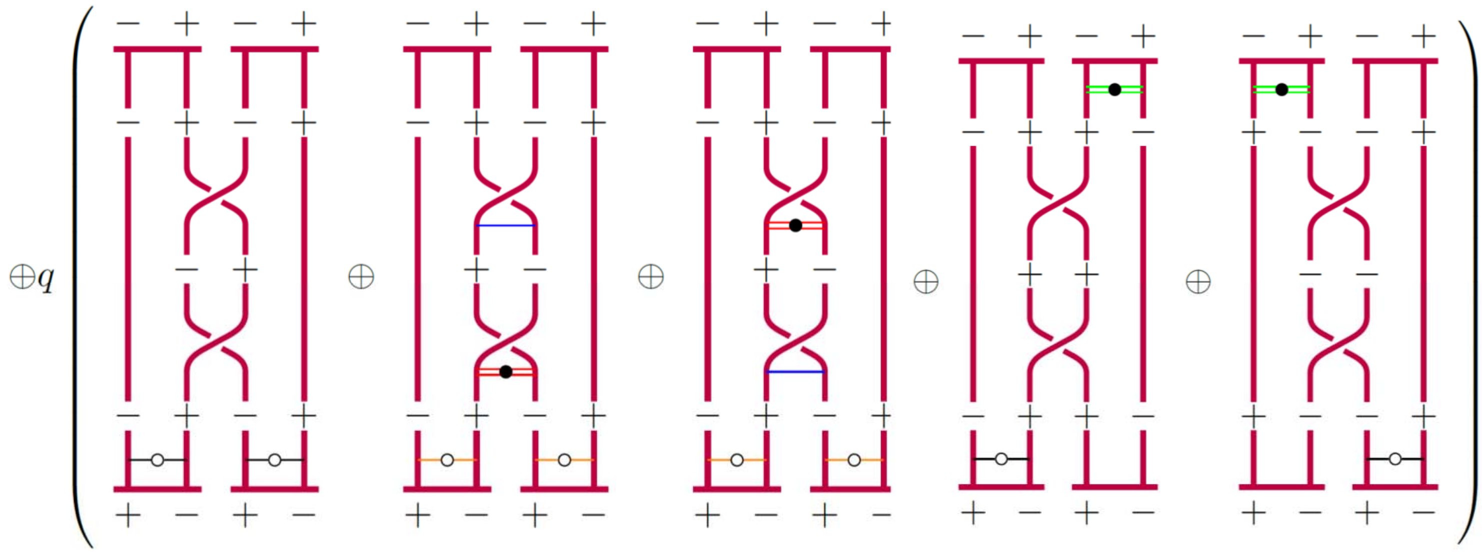
using Cecotti-Vafa tt* equations.

Example: Hopf Link





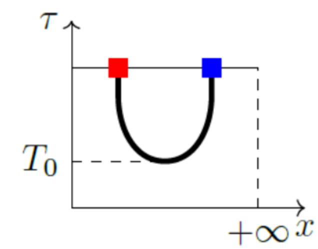
Example: Hopf Link



$$\mathcal{E}_1 = (0 \to \mathbb{Z}[\Psi_1] \to \mathbb{Z}[\Psi_2] \oplus \mathbb{Z}[\Psi_3] \to \mathbb{Z}[\Psi_4] \oplus \mathbb{Z}[\Psi_5] \to 0)$$

Differential obtained by counting ζ-instantons.

Example: $\langle \Psi_2 | \mathcal{Q}_\zeta | \Psi_1 \rangle = 1 \sim$



$$H^*(\mathcal{E}_1, \mathcal{Q}_\zeta) \cong \mathbb{Z}[\Psi_4 - \Psi_5]$$

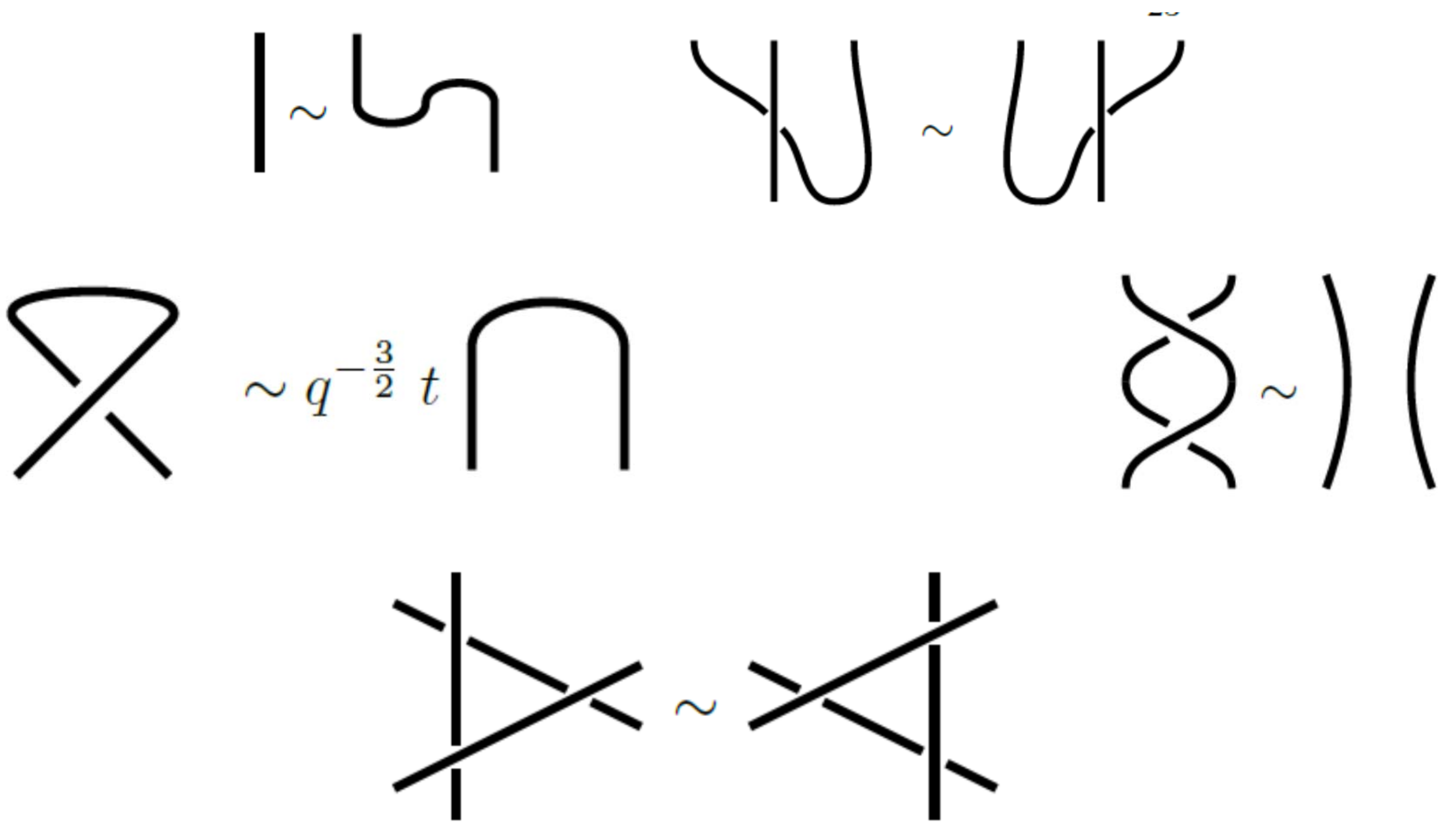
$$\mathcal{P}(q, t | \mathrm{Hopf}) = \left(\frac{q}{t} + \frac{t}{q}\right)\left(\frac{q^2}{t} + \frac{t}{q^2}\right)$$

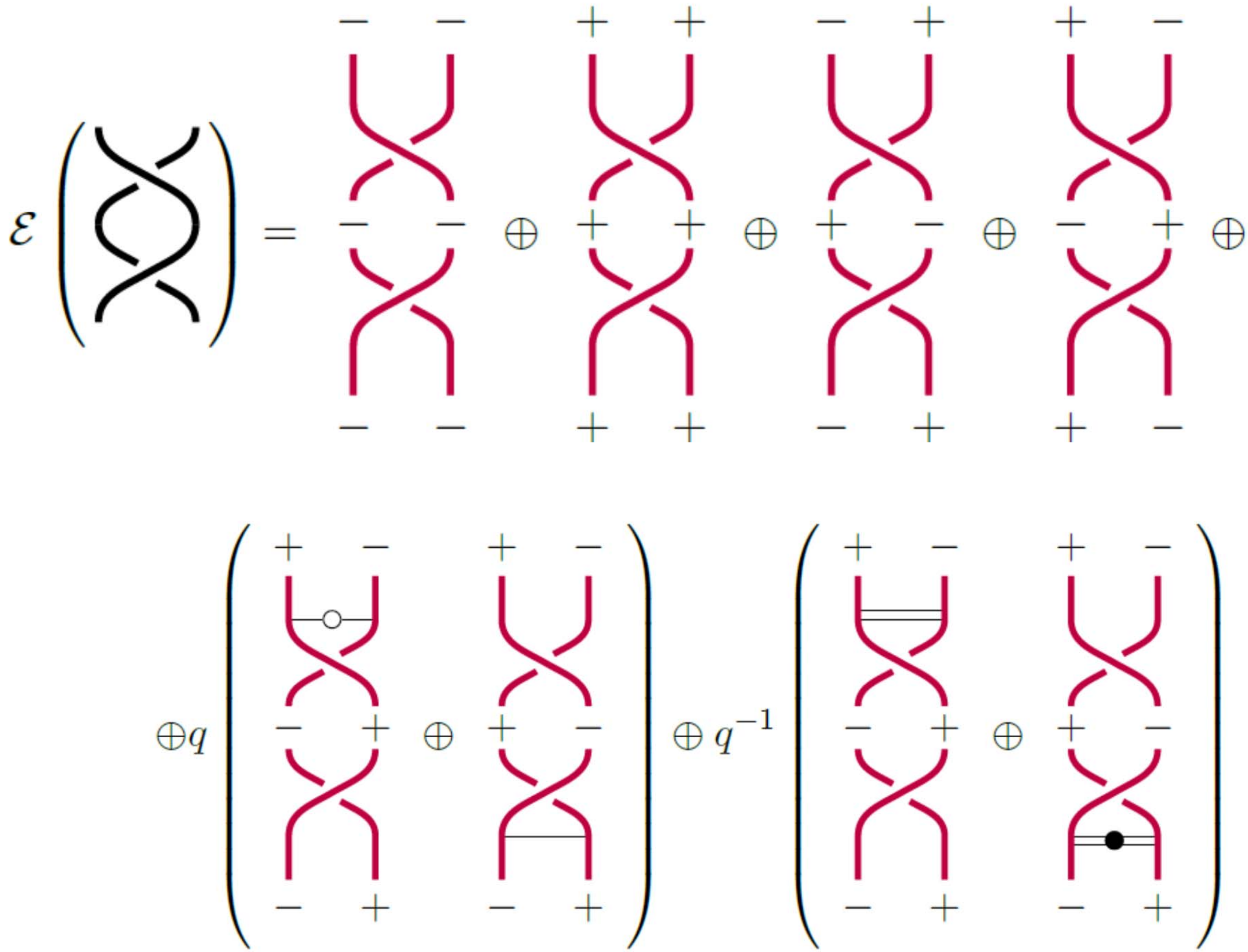
Reidemeister Moves

The complex depends on the link projection:

It does not have 3d symmetry

Need to check the homology DOES have 3d symmetry:




$\mathcal{E}$
$=$
$\oplus$
$\oplus q$
$\oplus\, q^{-1}$

Obstructions & Resolutions -1/2

In verifying invariance of the link complex up to quasi-isomorphism under RI and RIII we found an ***obstruction*** for the YYLG model due to walls of marginal stability and the non-simple connectedness of the target space.

Problem can be traced to the fact that in the YYLG $w_i \neq w_j \qquad w_i \neq z_a$

These problems are cured by the monopole model.

$$W = \sum_{i,a} k_a \log(w_i - z_a) - \sum_{i<j} \log(w_i - w_j)^2 + c \sum_i w_i$$

$$W = \sum_{i=1}^{m} \mathrm{Res}_{u=w_i} \frac{K(u)P(u)}{Q(u)} - \log P(w_i) + c w_i$$

$$Q(u) = \prod_{i=1}^{m} (u - w_i) \qquad K(u) = \prod_{i=1}^{n} (u - z_a)^{k_a}$$

Obstructions & Resolutions -2/2

Conclusion: YYLG does not give link homology, but MLG does.

Unpublished work of Manolescu reached the same conclusion for the YYLG.

M. Abouzaid and I. Smith have outlined a totally different strategy to recover link homology from MLG.

1. Review Derivation Of KS-WCF Using Framed BPS States

   (with D. Gaiotto & A. Neitzke, 2010, …)

2. Interfaces in 2d N=2 LG models & Categorical CV-WCF

   (with D. Gaiotto & E. Witten, 2015)

3. Application to knot homology

   (with D. Galakhov, 2016)

4. Semiclassical BPS States & Generalized Sen Conjecture

   (with D. van den Bleeken & A. Royston, 2015; D. Brennan, 2016)

5. Conclusion

The Really Hard Question

Data Determining A Framed BPS State In (Lagrangian) d=4 N=2 Theory

Action:

Compact semisimple Lie group G

Quaternionic representation $\mathcal{R}$

Mass parameters $m \in \mathrm{Adj}(G_{\mathrm{flavor}}) \otimes \mathbb{C}$

Line defect L: $\zeta = e^{i\vartheta}$ $[(P,Q)]$

Infrared: $u \in \mathcal{B}$ $\gamma \in \Gamma_u$

$$\overline{\underline{\mathcal{H}}}^{\mathrm{BPS}}(L,u,\gamma) = ???$$

For d=4 N=2 theories with a Lagrangian formulation at weak coupling there IS a quite rigorous formulation – well known to physicists…

$u \to \infty$ in a ``semiclassical chamber''

$$S = \int_{\mathbb{R}^4} \mathrm{Im}(\tau_0)\mathrm{Tr} F * F + \cdots$$

$$\mathrm{Im}(\tau_0) \to \infty$$

Method of collective coordinates:

Manton (1982); Sethi, Stern, Zaslow; Gauntlett & Harvey ; Tong; Gauntlett, Kim, Park, Yi; Bak, Lee, Yi; Bak, Lee, Lee, Yi; Stern & Yi; Manton & Schroers; Sethi, Stern & Zaslow; Gauntlett & Harvey ;Tong; Gauntlett, Kim, Park, Yi; Gauntlett, Kim, Lee, Yi; Bak, Lee, Yi; Lee, Weinberg, Yi; Tong, Wong;….

The Answer

One constructs a hyperholomorphic vector bundle over the moduli space of (singular) magnetic monopoles: $\mathcal{E} \to \underline{\overline{\mathcal{M}}}$

and Dirac-like operators $\mathbf{D}_{\mathcal{Y}}$ on : $\mathcal{S} \otimes \mathcal{E} \to \underline{\overline{\mathcal{M}}}$

$\mathrm{Ker}(\mathbf{D}_{\mathcal{Y}})$ is a representation of

$$T_{\mathbf{flavor}} \times T_{\mathbf{gauge}} \times SO(3)_{\mathbf{rot}} \times SU(2)_{\mathbf{R}}$$

$$\underline{\overline{\mathcal{H}}}^{\mathrm{BPS}}(L, u, \gamma) = \mathrm{ker}^{\gamma}_{L^2}\, \mathbf{D}_{\mathcal{Y}}$$

as representations of $SO(3)_{\mathbf{rot}} \times SU(2)_{\mathbf{R}}$

We use the only the data $G, \mathcal{R}, m, \zeta, P, Q, u, \gamma$

Exotic (Framed) BPS States

$\overline{\underline{\mathcal{H}}}^{\mathrm{BPS}}(L, u, \gamma)$ $\mathfrak{so}(3)_{\mathrm{rot}} \oplus \mathfrak{su}(2)_R$ -reps

Definition: ***Exotic BPS states***: States transforming nontrivially under $\mathfrak{su}(2)_R$

Conjecture [GMN]: $\mathfrak{su}(2)_R$ acts trivially: exotics don't exist.

Many positive partial results exist.

Cordova & Dumitrescu: Any theory with ``Sohnius'' energy-momentum supermultiplet (vanilla, so far…)

Geometrical Interpretation Of The No-Exotics Theorem - 2

Choose any complex structure on $\mathcal{M}$.

$$\mathcal{S} \cong \Lambda^{0,*}(T\mathcal{M}) \otimes K^{-1/2}$$

$$Q_3 + \mathrm{i}Q_4 \sim \bar{\partial}_{\mathcal{Y}} \qquad \mathcal{Y} \in \mathfrak{t}$$

$\mathfrak{su}(2)_R$ becomes ``Lefshetz $\mathfrak{sl}(2)$''

$$I^3|_{\Lambda^{0,q}} = \frac{1}{2}(q - N)\mathbf{1}$$

$$I^+ \sim \omega^{0,2}\wedge \qquad I^- \sim \iota(\omega^{2,0})$$

$$\mathcal{E} \otimes \mathbb{C} \cong \mathcal{W} \oplus \overline{\mathcal{W}}$$

Geometrical Interpretation Of The No-Exotics Theorem - 4

$$H^{0,q}_{L^2}(\bar{\partial}_{\mathcal{Y}}; \mathcal{W})$$

vanishes except in the middle degree q =N, and is primitive wrt ``Lefshetz $\mathfrak{sl}(2)$''.

$$\forall \mathcal{Y} \in \mathfrak{t}$$

SU(2) N=2* m $\rightarrow$ 0 recovers the famous Sen conjecture

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5 Conclusion

Conclusion -1/2

Lots of interesting & important questions remain about BPS indices:

- We still do not know the topological string partition function for a single compact CY3 with SU(3) holonomy !

- We still do not know the DT invariants for a single compact CY3 with SU(3) holonomy !

Nevertheless, we should also try to understand the spaces of BPS states themselves. Often it is useful to think of them as cohomology spaces of some complexes – and then these complexes satisfy wall-crossing – that ``categorification'' has been an important theme of this talk.

Conclusion – 2/2

A very effective way to address the (vanilla) BPS spectrum is to enhance the zoology to include new kinds of BPS states associated to defects.

As illustrated by knot theory and the generalized Sen conjecture, understanding the vector spaces of (framed) BPS states can have interesting math applications.