Some Questions Of Possible Interest To This Simons Collaboration

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Outline

Part 1:Time Reversal In Chern-Simons-Witten Theory

Part 2: Three Questions About SYM & Four-manifold Invariants

Part 1

Time Reversal In
Chern-Simons-Witten Theory

When does 3d Chern-Simons-Witten theory have a time reversal symmetry?

General theory based on compact group G and a ``level'' $k \in H^4(BG;\mathbb{Z})$

Which (G, k) give T-reversal invariant theories?

Related: When does Reshetikhin-Turaev-Witten topological field theory factor through the unoriented bordism category?

Some nontrivial examples of T-invariant CSW theories appeared in several recent papers

[Seiberg & Witten 2016; Hsin & Seiberg 2016; Cordova, Hsin & Seiberg]

$PSU(N)_N$ $USp(2N)_N$ $PSU(2)_6$

$U(N)_{N,2N}$ $SO(N)_N$ $SU(N)_2$ $\left(-\frac{1}{N}\right) = 1$

But there is no systematic understanding.

With my student Roman Geiko we have recently carried out a systematic study for

Spin Chern-Simons Theory with torus gauge group $G \cong U(1)^r$

$$S = \frac{1}{4\pi} \int K_{IJ} \, A_I \, d \, A_J$$

K_{IJ} : $r \times r$ nondegenerate, integral symmetric matrix: determines integral lattice L

Classical T-reversal:

$\exists \; U \in GL(r, \mathbb{Z})$ such that

$$UKU^{tr} = -K$$

(Note: $\sigma(L) = 0$)

But there can be quantum T-reversal symmetries not visible classically.

Rank 2 examples studied by Seiberg & Witten; Delmastro & Gomis

The quantum theory does not depend on all the details of L

What *does* it depend on?

Finite Abelian group $\mathcal{D}(L) := L^\vee/L$

a.k.a ``group of anyons''   a.k.a. ``group of 1-form symmetries''

Quadratic Refinement (spin of anyons) :

$$q_W(x) = \frac{1}{2}(\tilde{x}, \tilde{x} - W) + \frac{1}{8}(W, W) \ mod\, \mathbb{Z}$$

$$\frac{1}{\sqrt{|\mathcal{D}(L)|}} \sum_{x \in \mathcal{D}(L)} e^{2\pi i\, q_W(x)} = e^{2\pi i \frac{\sigma(L)}{8}}$$

Theorem
[Belov & Moore; Freed,Lurie,HopkinsTeleman]

The quantum theory only depends on the equivalence class of the triple $(\mathcal{D}, q, \bar{\sigma})$

$$q: \mathcal{D} \to \mathbb{R}/\mathbb{Z} \qquad \bar{\sigma} \in \mathbb{Z}/24\mathbb{Z}$$

$$\frac{1}{\sqrt{|\mathcal{D}|}} \sum_{x \in \mathcal{D}} e^{2\pi i\, q(x)} = e^{2\pi i \frac{\bar{\sigma}}{8}}$$

Conversely, every such triple arises from some torus CSW theory

Equivalence of triples

$$(\mathcal{D}, q, \bar{\sigma}) \cong (\mathcal{D}', q', \bar{\sigma})$$

$\exists$ isomorphism $\quad f : \mathcal{D} \to \mathcal{D}'$

$$\exists \ \ \Delta' \in \mathcal{D}'$$

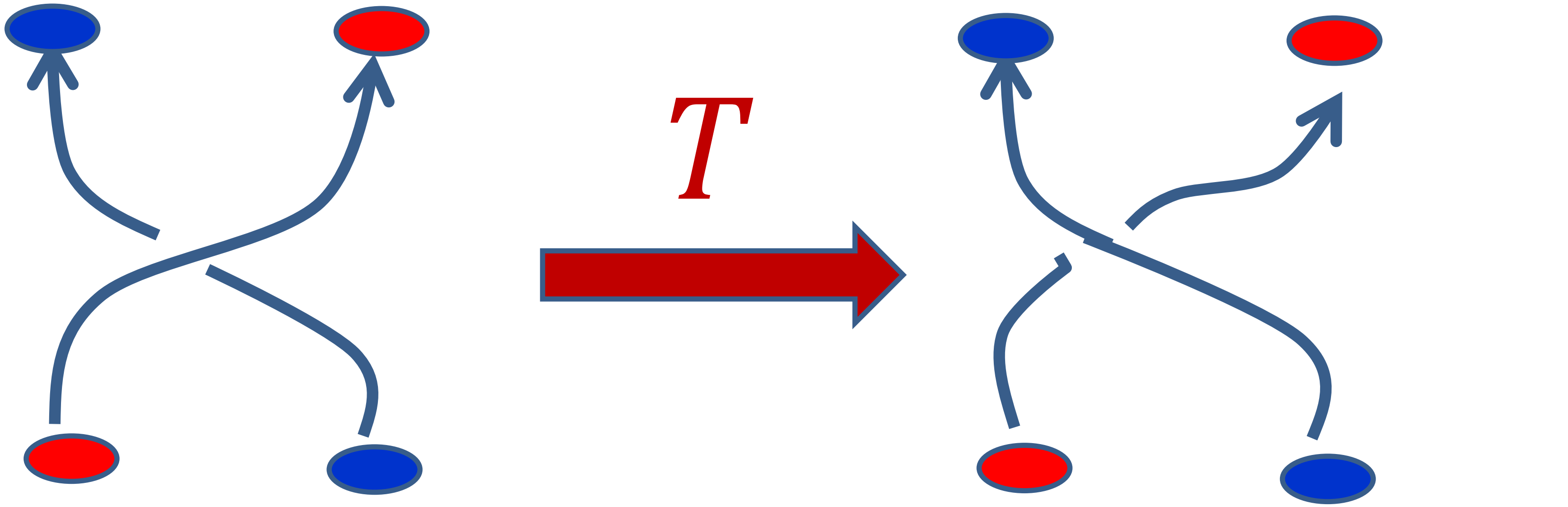
$$q(x) = q'(f(x) + \Delta')$$

T-Reversal Criterion

$$[(\mathcal{D}, q, \bar{\sigma})] = [(\mathcal{D}, -q, -\bar{\sigma})]$$

q: Determines the spin of anyons

b: Determines the braiding of anyons



Simpler Problem: The Witt Group (1936)

$$b(x, y) = q(x + y) - q(x) - q(y) + q(0)$$

Throw away $q, \bar{\sigma}$ and just keep b.

Classify $[(\mathcal{D}, b)]$

$$[(\mathcal{D}_1, b_1)] + [(\mathcal{D}_2, b_2)] := [(\mathcal{D}_1 \oplus \mathcal{D}_2, b_1 \oplus b_2)]$$

Abelian monoid $\mathcal{DB}$

$$\mathcal{DB} = \bigoplus_p \ \mathcal{DB}_p$$

Odd p: $\mathcal{DB}_p$ is generated by forms on $\mathbb{Z}/p^r\mathbb{Z}$

X_{p^r}: $b(1,1) = p^{-r}$ Y_{p^r}: $b(1,1) = \theta p^{-r}$

θ: Quadratic nonresidue modulo p^r

$p = 2$ Many generating forms:

$$A_{2^r}, B_{2^r}, C_{2^r}, \ldots \ , F_{2^r}$$

Submonoid $\mathcal{Spl}$ Split forms:

$$\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$$

$$\mathcal{D}_1 = \mathcal{D}_1^{\perp}$$

$$\mathcal{Witt} \quad := \quad \mathcal{DB}/\mathcal{Spl}$$

Abelian group whose structure is known.

Wall, Miranda, Kawauchi & Kojima

determine relations on the generators

$$\mathcal{Witt} \cong \bigoplus_p \mathcal{Witt}_p$$

p odd: $\quad \mathcal{Witt}_p \cong \bigoplus_{k \geq 1} \mathcal{W}_p^k$

$\mathcal{W}_p^k \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \qquad \left(-\frac{1}{p}\right) = (-1)^{\frac{p-1}{2}} = 1$

$\mathcal{W}_p^k \cong \mathbb{Z}_4 \qquad \left(-\frac{1}{p}\right) = (-1)^{\frac{p-1}{2}} = -1$

$$\mathcal{Spl} \subset \mathcal{DB}^T := \{ [\mathcal{D}, b] = [\mathcal{D}, -b] \} \subset \mathcal{DB}$$

Theorem 1: The classes in $\mathcal{DB}^T$ descend to order 2 elements of $\mathcal{Witt}$ and all of order 2 elements are represented by a T-invariant bilinear form.

Generalization to quadratic refinements is nontrivial.

$$\mathcal{D} = \mathbb{Z}/2\mathbb{Z}$$

$$b(x,y) = \frac{xy}{2} \ mod\, 1$$ is T-invariant

$$q(x) = \frac{x^2}{4} \ mod\, 1$$ is not T-invariant

$$\hat{q}(x) = \frac{x^2}{4} - \frac{1}{8}$$ and $$\hat{q}(x) = \frac{x^2}{4} + \frac{3}{8}$$

T-invariant because $\exists\, \Delta$: $\hat{q}(x+\Delta) = -\hat{q}(x)$

Higher Gauss Sums

$$\tau_n(q) := \sum_{x \in \mathcal{D}} e^{2\pi i n\, q(x)}$$

Theorem 2: $(\mathcal{D}, q)$ is T-invariant iff $\tau_n(q)$ are real for all $\mathrm{n} = 1,2,3, \ldots$

$\sigma = 0\ mod\ 8$ refines uniquely to T-invariant $\bar{\sigma} = 0$ mod 24

$\sigma = 4\ mod\ 8$ refines uniquely to T-invariant $\bar{\sigma} = 12$ mod 24

Theorem 3: Given a T-invariant $(\mathcal{D}, b)$ and $\sigma \in \{0,4\}\ mod\ 8$, there is, up to isomorphism, exactly one T-invariant quadratic refinement $(\mathcal{D}, \hat{q})$ such that the phase of τ_1 is $e^{2\pi i\frac{\sigma}{8}}$

$Quad - Witt$ Groups

$\mathcal{QW}$: Mod out the monoid of $(\mathcal{D}, q)$ by $(\mathcal{D}_1, q_1) \sim (\mathcal{D}_2, q_2)$ if there are isotropic subgroups $\mathcal{H}_i \subset \mathcal{D}_i$ such that

$$(\mathcal{H}_1^{\perp}/\mathcal{H}_1 \, , q_1) \cong (\mathcal{H}_2^{\perp}/\mathcal{H}_2 \, , q_2)$$

Theo J-F: There are 2-torsion elements of $\mathcal{QW}$ represented by non-T-invariant $(\mathcal{D}, q)$

$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z}$ $\qquad \frac{2x^2}{9} - \frac{2y^2}{81} \quad mod\, 1$

$\mathcal{T} - \mathcal{W}itt$ Groups

$\mathcal{T} - \mathcal{W}itt$: Mod out the submonoid of T-invariant $(\mathcal{D}, q)$ by $(\mathcal{D}_1, q_1) \sim (\mathcal{D}_2, q_2)$ if there are isotropic subgroups $\mathcal{H}_i \subset \mathcal{D}_i$ *(invariant under a T-symmetry)*, such that

$$(\mathcal{H}_1^{\perp}/\mathcal{H}_1 \, , q_1) \cong (\mathcal{H}_2^{\perp}/\mathcal{H}_2 \, , q_2)$$

$$\mathcal{T} - \mathcal{W}itt \cong Ord_2(\mathcal{QW})$$

Interfaces

$\mathcal{T} - \mathcal{W}itt$ equivalence detects the existence of a T-reversal symmetric interface between T-invariant theories.

Conjecture for the general case:
(TQFT, not spin-TQFT)

$$(G,k) \to CSW(G,k) \to \quad MTC(G,k)$$

Definition [Lee & Tachikawa; Kong & Zhang]: The time reversal of an MTC $\mathcal{C}$ with braiding $B_{x,y}: x \otimes y \to y \otimes x$ and ribbon structure $\theta_x: x \to x$ is the MTC $\mathcal{C}^{rev}$ with

$$B^{rev}_{x,y} := B^{-1}_{y,x} \qquad \theta^{rev}_x := \theta^{-1}_x$$

A CSW theory is time reversal invariant if there is an equivalence of MTC's

$$MTC(G,k)^{rev} \cong MTC(G,k)$$

Conjectural T-Invariance Condition

$$\tau_n(\mathcal{C}) := \sum_x d_x^2 \theta_x^n \ = (ST^nS)_{00}$$

[Ng, Schopieray, Wang 2018;
Kaidi, Komargodski,Ohmori,Seifnashri, Shao 2021]

Conjecture 1: An MTC is T-invariant iff all the higher Gauss sums are real for all $n = 1,2,3, \ldots$

The examples of Seiberg et. al. satisfy this condition.

Is there a Witt group interpretation?

There is a mathematical notion of a Witt group of (nondegenerate) braided fusion categories.

[Davydov, Müger, Nikshych, Ostrik 2010]

$\mathcal{C}_1 \sim \mathcal{C}_2$ if there exist fusion categories $\mathcal{D}_1$ and $\mathcal{D}_2$ such that

$$\mathcal{C}_1 \otimes Z(\mathcal{D}_1) \cong \mathcal{C}_2 \otimes Z(\mathcal{D}_2)$$

Interpretations

It is always true that $\mathcal{C} \otimes \mathcal{C}^{rev} \cong Z(\mathcal{D})$

So T-reversal invariant MTC's define order 2 elements of the Witt group of fusion categories.

Result of Freed & Teleman $\Rightarrow$ $\mathcal{C} \otimes \mathcal{C}$ admits a gapped topological boundary condition

Can interpret with topological interfaces.

C. Schweigert: There is a 1-morphism in the Morita 2-category BrTens

But….

In the case of Abelian group fusion rules (``pointed MTC'') $\mathcal{Witt}_{fus.cat.}$ becomes the group $\mathcal{QW}$.

We saw above that being order 2 in $\mathcal{QW}$ is NOT a criterion for T-invariance!

So T-reversal invariance is NOT the same as order 2 in the Witt group! (Thanks: Theo Johnson-Freyd)

Conjecture 2: There is an analog of the $\mathcal{T}-\mathcal{Witt}$ group for fusion categories consisting of T-reversal invariant MTC's with T-reversal invariant boundary conditions, and this group will be isomorphic to the subgroup of $\mathcal{Witt}_{fus.cat.}$ of elements of order two.

Part 2

Three Questions About SYM & Four-manifold Invariants

First Question

Topological Twisting

(Some new comments based on Manschot-Moore 2021, and some discussions with Dan Freed)

Topological twisting of d=4 N=2 field theories leads to diffeomorphism invariants of smooth compact oriented four-manifolds.

Basic Example: ``pure'' SU(2) N=2 SYM

Super-Poincaré algebra: $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$

Algebra of supertranslations $\mathfrak{g}^1$
(and fields in the ``vectormultiplet) are in representations of a global symmetry group G^0 with Lie algebra $\mathfrak{g}^0$

$$G^0 = (SU(2)_+ \times SU(2)_- \times SU(2)_R)/ \mathcal{Z}$$

$$\mathcal{Z} = \langle \, (-1, -1, -1 \,) \rangle \cong \mathbb{Z}_2$$

Background fields of untwisted theory: Riemannian metric and orientation on X

…. and a G^0-connection

Metric and orientation give a reduction of structure group of TX to

$$H := (SU(2)_+ \times SU(2)_-)/\langle(-1,-1)\rangle$$

with connection ∇^{LC}

There is a homomorphism $\rho\colon H \to G^0$

$$[(u_1, u_2)] \to [(u_1, u_2, u_1)]$$

(Improved) Definition of topological twisting: A ``topologically twisted theory'' is the untwisted theory with a choice of background fields so that there is a reduction of structure group and background fields to H, ∇^{LC} under ρ

Remarkable fact (Witten 1988): With the above choice of background fields the partition function and correlation functions of certain ``operators'' are independent of ∇^{LC}.

They are therefore smooth invariants of 4-folds

Other Theories

There are infinitely many other d=4 N=2 field theories.

Twisted versions might, or might not, teach us new things about differential topology of four-manifolds.

What are the (topological) background fields of the other twisted theories?

Example: $SU(2), N = 2^*$ Symmetry group is

$$G^0 = \left(SU(2)_+ \times SU(2)_- \times SU(2)_R \times U(1) \right) / \mathcal{Z}$$

$$\mathcal{Z} = \langle (-1,-1,-1,-1) \rangle \cong \mathbb{Z}_2$$

There is ***NO*** homomorphism from the structure group of TX to G^0

(compatible with constraints on the morphism of Lie algebras)

There ***IS*** a homomorphism $Spin^c(4) \to G^0$

The twisted theory correlation functions are independent of the $Spin^c$ *connection* but do depend nontrivially on the $Spin^c$ *structure* on the 4-fold [Manschot & Moore]

One can analyze by hand the topological data for all the Lagrangian theories (wip: Ranveer Singh)

Question:

What about non-Lagrangian theories?

What is the background topological data for twisting the general class S theory $T[\mathfrak{g}, C_{g,n}, D]$?

Second Question:
Invertible Theories And Orientation

Twisted SYM & Anomaly Theories

The path integral for topologically twisted Lagrangian theories localizes to intersection theory on moduli space of the nonabelian Seiberg-Witten equations

(instanton moduli space is a special case)

How Twisted Lagrangian Theories Generalize Donaldson Invariants

$$Z(S) = \langle e^{\mathcal{O}(S)} \rangle_{\mathcal{T}} = \int_{\mathcal{M}} e^{\mu(S)} \mathcal{E}(\mathcal{V})$$

But now $\mathcal{M}$: is the moduli space of:

$$F^{+} = \mathcal{D}(M, \bar{M}) \qquad \gamma \cdot D\, M = 0$$

$$M \in \Gamma(W^{+} \otimes V)$$

W^{+}: Rank 2 ``spin'' bundle; V depends on matter rep

``Nonabelian Seiberg-Witten equations''

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

Defining the integral over $\mathcal{M}$ requires a choice of orientation

Orientability should be determined by the mod-two index of the deformation complex

$\sim$ mod two index of Dirac coupled to relative spin bundle [Atiyah-Hitchin-Singer]

+ …

View the determinant bundle of the deformation complex as the (real) state space of a 5d invertible theory

For the original case of Donaldson theory, with rank one gauge group the 5d invertible theory is

$$\int w_2(P)\, Sq^1\, w_2(TX)$$

P: Principal $SO(3)$ bundle over X for the gauge fields

(Related to remarks of Kapustin & Thorngren 2017; Cordova & Dumitrescu 2018; Wang, Wen & Witten 2019)

An orientation is a trivialization of this invertible theory.

This nicely summarizes some facts about the relation of SYM and Donaldson invariants:

Sign of the Donaldson invariants depends on an integral lift modulo 4 of $w_2(P)$

(as in Donaldson & Kronheimer's book)

OR on an integral lift modulo 4 of $w_2(TX)$

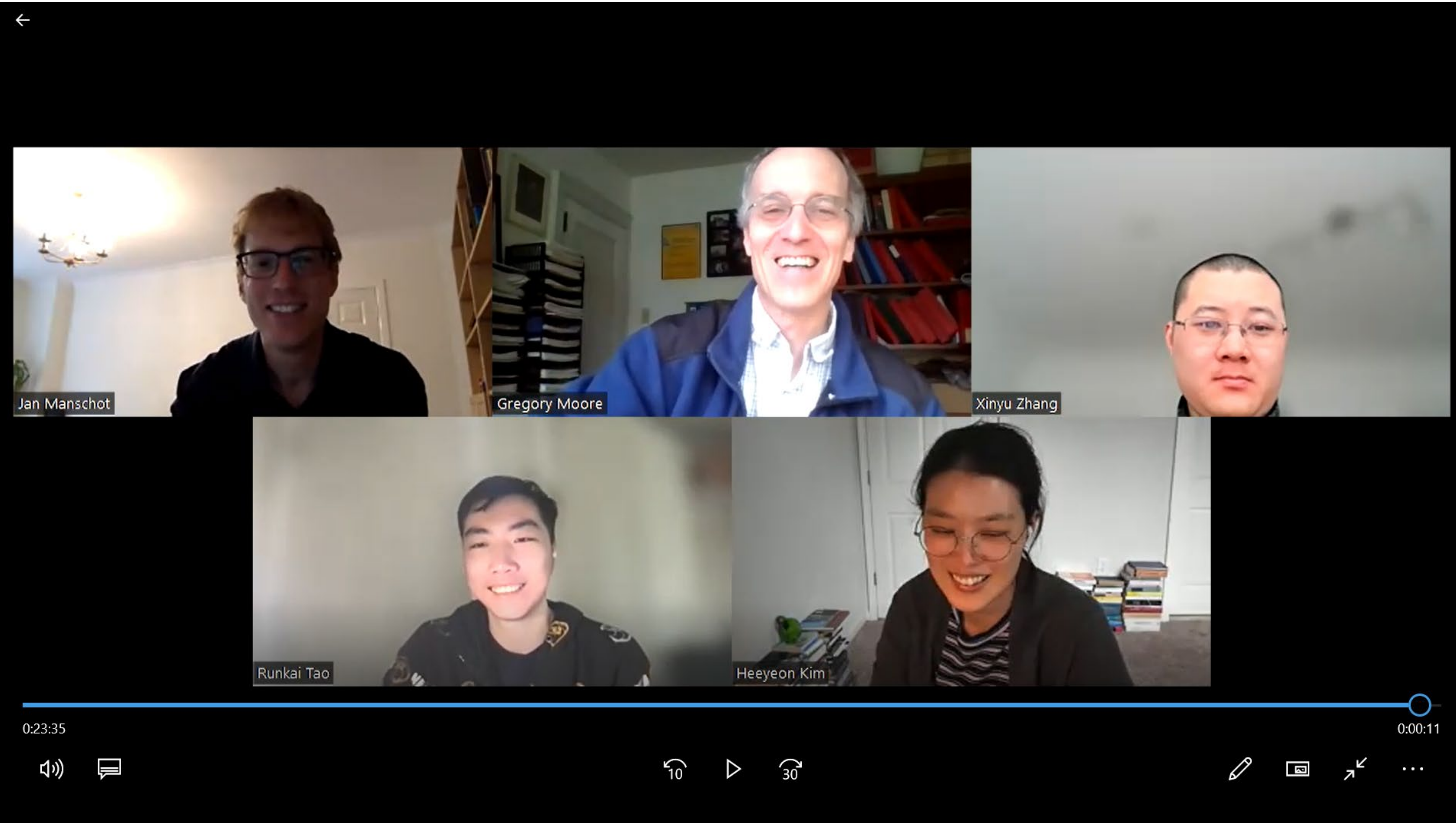
(as in Witten, 1994)

With an explicit counterterm between these choices (Moore & Witten 1997)

Question:
Is there a useful description of the analogous 5d invertible theory for the moduli space of the nonabelian Seiberg-Witten equations for general compact group and quaternionic representation?

Third Question: Puzzle About ``K-theoretic Donaldson Invariants’’

``K-Theoretic Donaldson Invariants''



Five Dimensions

Partial Topological Twist of 5d SYM on $X \times S^1$

$$Q^2 = \partial_t$$

Topological on X $\Rightarrow$ Can shrink X $\Rightarrow$
Describe the twisted theory in terms of SQM with target space the moduli space of instantons:

But $\mathcal{M}$ is not spin in general,
so the theory will be anomalous

Find a suitable ``line bundle'' $\mathcal{L}$ so that $S^{+}_{\mathcal{M}} \otimes \mathcal{L}$ exists

Chern-Simons Observables

$U(1)_{inst}$ symmetry with current $J = TrF^2$

Couple to background gauge field $\mathrm{A_{bck}}$ $n := c_1(P_{inst}) \in H^2(X, \mathbb{Z})$

$$\mathcal{O}(n) = \int_{\Sigma(n)\times S^1} Tr\left(AdA + \frac{2}{3}A^3\right) + \cdots$$

$$= \int_{X\times S^1} F(A_{bck}) \wedge Tr\left(AdA + \frac{2}{3}A^3\right)$$

+ susy completion

Introduces a ``line bundle'' $L(n) \to \mathcal{M}$ in the SQM

Conjecture: For suitable n $S^{+}_{\mathcal{M}} \otimes L(n)$ exists

Evidence: One can show

X admits an acs $\Rightarrow$ $\mathcal{M}_k$ is spin-c

$D_{L(n)}$: Dirac operator coupled to $L(n)$

At least formally the path integral should compute

$$Z(\mathcal{R}, n) = \sum_{k=0}^{\infty} \mathcal{R}^{\frac{d_k}{2}} \; Tr_{\mathcal{H}_k}\{(-1)^F \exp(-\mathcal{R}\, D^2_{L(n)} \;)\}$$

$$d_k = \dim_{\mathbb{R}} \mathcal{M}_k$$

$$Z(\mathcal{R}, n) = \sum_{k=0}^{\infty} \mathcal{R}^{\frac{d_k}{2}} \; Tr_{\mathcal{H}_k}\{(-1)^F \exp(-\mathcal{R}\, D^2_{L(n)}\ )\}$$

$$\mathcal{R} = R\,\Lambda$$

Λ dimensional scale in the physical theory

$$\mathcal{R}^4 = \exp\left[-8\,\pi^2 \frac{R}{g^2_{5d,YM}} + i\,\theta\right]$$

$$m_{inst.part.} = \frac{1}{R} \log \mathcal{R}^2$$

$$Tr_{\mathcal{H}_k}\{(-1)^F \exp(-\mathcal{R}\, D_{L(n)}^2\ )\}$$

In good cases, this is the index of the Dirac operator $D_{L(n)}$

$\Rightarrow$ ``K-theoretic Donaldson invariants''

All this should generalize to (anomaly-free) 6d SYM theories on $X \times \mathbb{E}$

$$Index\left(D_{L(n)}\right) \to Ell(\sigma(\mathcal{M}_k))$$

Five Dimensions

$$Z(\mathcal{R}, n)`` = '' \sum_{k=0}^{\infty} \mathcal{R}^{d_k/2} \int_{\mathcal{M}_k} ch\big(L(n)\big)\, \hat{A}(\mathcal{M}_k)$$

[Nekrasov, 1996; Losev, Nekrasov, Shatashvili, 1997]

Using both the Coulomb branch integral (a.k.a. the U-plane integral) and, independently, localization techniques, we make contact with the work of mathematicians

2006:

K-THEORETIC DONALDSON INVARIANTS VIA INSTANTON COUNTING

LOTHAR GÖTTSCHE, HIRAKU NAKAJIMA, AND KŌTA YOSHIOKA

To Friedrich Hirzebruch on the occasion of his eightieth birthday

Abstract. In this paper we study the holomorphic Euler characteristics of determinant line bundles on moduli spaces of rank 2 semistable sheaves on an algebraic surface X, which can be viewed as K-theoretic versions of the Donaldson invariants. In particular if X is a smooth projective toric surface, we determine these invariants and their wall-crossing in terms of the K-theoretic version of the Nekrasov partition function (called 5-dimensional supersymmetric Yang-Mills theory compactified on a circle in the physics literature). Using the results of [43] we give an explicit generating function for the wall-crossing of these invariants in terms of elliptic functions and modular forms.

2019:

VERLINDE FORMULAE ON COMPLEX SURFACES I: K-THEORETIC INVARIANTS

L. GÖTTSCHE, M. KOOL, AND R. A. WILLIAMS

Abstract. We conjecture a Verlinde type formula for the moduli space of Higgs sheaves on a surface with a holomorphic 2-form. The conjecture specializes to a Verlinde formula for the moduli space of sheaves. Our formula interpolates between K-theoretic Donaldson invariants studied by the first named author and Nakajima-Yoshioka and K-theoretic Vafa-Witten invariants introduced by Thomas and also studied by the first and second named authors. We verify our conjectures in many examples (e.g. on K3 surfaces).

Using the two physical techniques we derived (compatible) results

Differ from GNY!

Agree with GNY!

(Suitably interpreted.)

This raises our main puzzle…

Total partition function is a sum of two terms

$$Z^J(\mathcal{R}, n) = \Phi^J(\mathcal{R}, n) + Z^J_{SW}(\mathcal{R}, n)$$

$\Phi^J(\mathcal{R}, n)$: 4d Coulomb branch integral

Z^J_{SW} : Contribution of SW invariants

One can deduce Z^J_{SW} from Φ^J

For $b_2^+ = 1$ there is metric dependence through the period point J

$$J \in H^2(X, \mathbb{R}): \quad J = *J \ \ \& J^2 = 1 \ \& J \in Positive\ LC$$

$$b_2^+ > 1 \Rightarrow \ \Phi^J = 0$$

SW special Kahler geometry is subtle

For 5d SYM gauge group of rank 1:
Coulomb branch = $\mathbb{C}$

a: cylinder valued

$$\mathcal{F} \sim R^{-2} Li_3(e^{-2Ra}) + \cdots$$

$$+Instanton\ corrections\ (e^{Ra})$$

[Nekrasov, 1996]

$$U = \langle\, Pexp \oint_{S^1} \left(\sigma + i\, A_{5d,ym}\right) \,\rangle$$

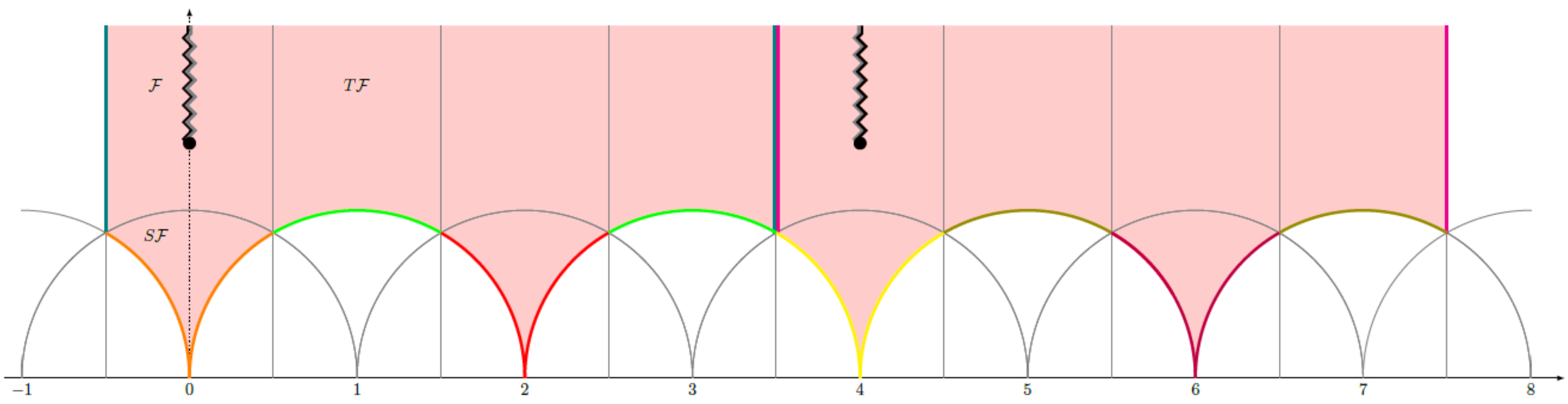
Modular Parametrization Of U −plane

The Coulomb branch is a branched double cover of the modular curve for $\Gamma^0(4)$

$$\left(\frac{U}{R}\right)^2 + u(\tau) = 8 + 4(\mathcal{R}^2 + \mathcal{R}^{-2})$$

$$u(\tau) = \frac{\vartheta_2(\tau)^2}{\vartheta_3(\tau)^2} + \frac{\vartheta_3(\tau)^2}{\vartheta_2(\tau)^2}$$

Hauptmodul for $\Gamma^0(4)$



$$\Phi^J(\mathcal{R}, n) = \int_{\mathcal{F}} d\tau d\bar{\tau}\, \nu\, C^{n^2}\ \ \Psi^J\left(\tau, \frac{n}{2}\,\zeta\right)$$

$$\nu(\tau, \mathcal{R}) = \frac{\vartheta_4^{13-b_2}}{\eta^9}\ \ \frac{1}{\sqrt{1 - 2\,\mathcal{R}^2 u(\tau) + \mathcal{R}^4}}$$

$C(\tau, \mathcal{R})$ Suitably modular invariant and holomorphic ``contact term''

$$\Psi^J(\tau, z) = \sum_{k \in H^2(X,\mathbb{Z})} \left(\frac{\partial}{\partial \bar{\tau}} E_k^J \right) q^{-\frac{k^2}{2}} e^{-2\pi i\, k\cdot z} \, (-1)^{k\cdot K}$$

$$E_k^J = Erf\left(\sqrt{Im\tau} \, \left(k + \frac{Im\, z}{Im\, \tau} \right) \cdot J \right)$$

$$z \to \frac{n}{2} \zeta(\tau, \mathcal{R})$$

Not holomorphic. Metric dependent .

Formally: A total derivative:

$$\frac{\partial}{\partial \bar{\tau}} \hat{G} = \Psi$$

Measure As A Total Derivative

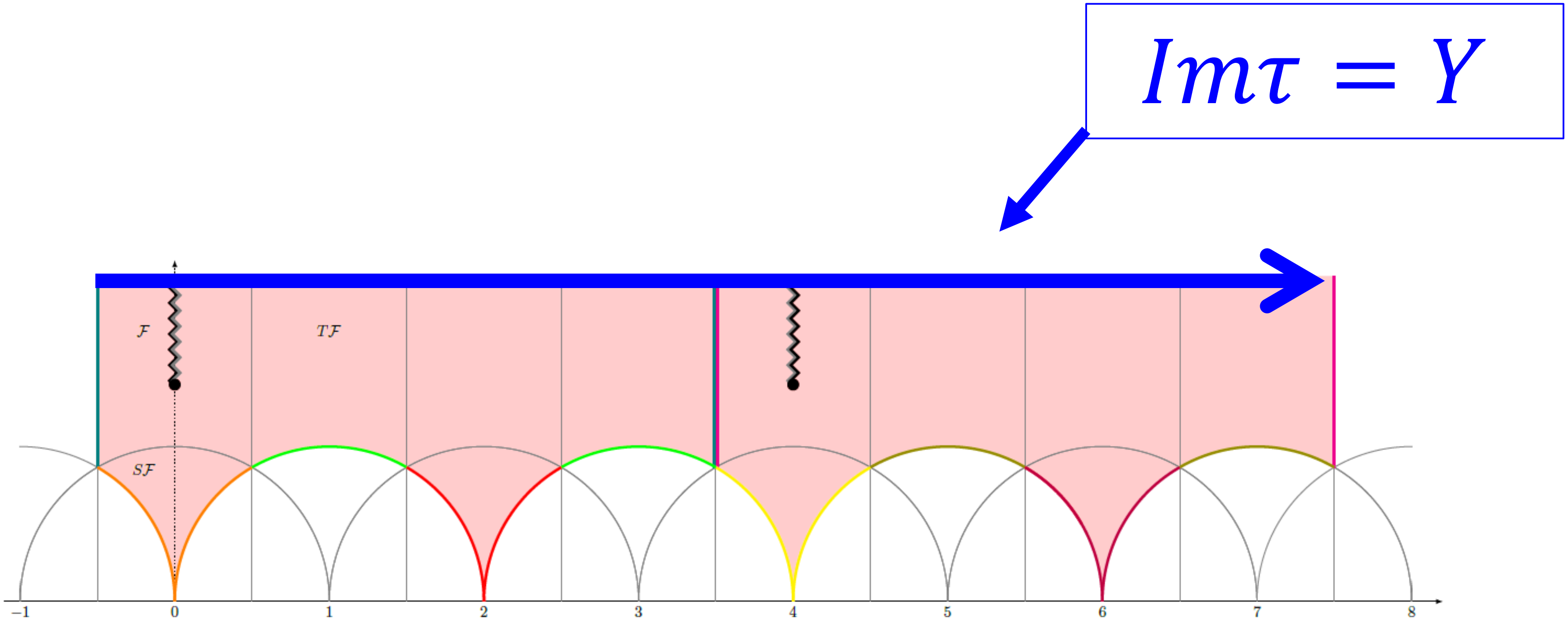
$$\Phi^J(\mathcal{R}, n) = \int_{\mathcal{F}} d\tau d\bar{\tau}\, \nu\, C^{n^2} \;\; \Psi^J\left(\tau, \frac{n}{2}\zeta\right)$$

$$\Omega = d\,\Lambda \qquad\qquad \Lambda = d\tau\, \mathcal{H}\;\; \hat{G}$$

For a suitably modular invariant and nonsingular $\hat{G}(\tau, \bar{\tau})$

$$\frac{\partial}{\partial\bar{\tau}}\hat{G} = \Psi$$

(It can be hard to find explicit formulae for $\hat{G}$: one needs the theory of mock modular forms, and their generalizations.)



$$\Phi^J(n,\mathcal{R}) = \lim_{Y\to\infty} \int d\tau_1\, \mathcal{H}\, \hat{G}\,\Big|_{\tau=\tau_1+i\,Y}$$

Explicit Results

$$X = \mathbb{CP}^2 \qquad \Phi(n,\mathcal{R}) = \left[\, \nu(\tau,\mathcal{R})\; C(\tau,\mathcal{R})^{n^2}\; G(\tau,\mathcal{R}) \right]_{q^0}$$

$$G(\tau,\mathcal{R}) = -\frac{e^{i\pi n \frac{\zeta(\tau,\mathcal{R})}{2}}}{\vartheta_4(\tau)} \sum_{\ell\in\mathbb{Z}} (-1)^{\ell}\; \frac{q^{\frac{\ell^2}{2}-\frac{1}{8}}}{1 - e^{i\pi n\,\zeta(\tau,\mathcal{R})} q^{\ell-\frac{1}{2}}}$$

Wall Crossing Formula:

$$\Phi^{J} - \Phi^{J'} = \left[\, \nu\, C^{n^2} \Theta^{J,J'} \right]_{q^0}$$

If we take these formula literally, we get results that are very different from GNY

We get finite Laurent polynomials in $\mathcal{R}$ with terms involving negative powers of $\mathcal{R}$

It looks nothing like:

$$Z(\mathcal{R}, n) = \sum_{k=0}^{\infty} \mathcal{R}^{d_k/2} \int_{\mathcal{M}_k} e^{c_1(L(n))} \hat{A}(\mathcal{M}_k)$$

$\nu, C, G, \Theta^{J,J'}$ are functions of τ and of $\mathcal{R}$

Subtle order of limits: $\mathcal{R} \to 0$ vs. $\Im\tau \to \infty$

Example: $u(\tau) \sim \frac{1}{8} q^{-\frac{1}{4}} + \frac{5}{2} q^{\frac{1}{4}} - \frac{31}{4} q^{\frac{3}{4}} + \mathcal{O}\left(q^{\frac{5}{4}}\right)$

$$\nu(\tau, \mathcal{R}) = \frac{\vartheta_4^{13-b_2}}{\eta^9} \frac{1}{\sqrt{1 - 2\,\mathcal{R}^2 u(\tau) + \mathcal{R}^4}}$$

$$\Phi(n,\mathcal{R}) = \Big[\ \nu(\tau,\mathcal{R})\ \ C(\tau,\mathcal{R})^{n^2}\ \ G(\tau,\mathcal{R})\Big]_{q^0}$$

$$\Phi^{J} - \Phi^{J'} = \Big[\nu\, C^{n^2}\Theta^{J,J'}\Big]_{q^0}$$

If we *first* expand the expressions in $[\ldots.]$ in $\mathcal{R}$ around $\mathcal{R} = 0$ *then* take the constant q^0 term at each order in $\mathcal{R}$ we find remarkable and nontrivial agreement with results in GNY.

Did we make a technical mistake?

Probably not:

Using toric localization and the 5d instanton partition function we derived exactly the same formula for wall-crossing @ ∞

Agreement with GNY
– with an admittedly ad hoc interpretation of the integral –
is extremely nontrivial.

Moreover, using the wall-crossing behavior of $\Phi^J(\mathcal{R}, n)$ at the strong coupling cusps allows one to ***derive*** Z_{SW}^J $\Rightarrow$ partition function for $b_2^+ > 1$

$$G(\mathcal{R}, n) = \frac{2^{2\chi+3\,\sigma-\chi_h}}{(1-\mathcal{R}^2)^{\frac{1}{2}n^2+\chi_h}} \sum_c SW(c) \left(\frac{1+\mathcal{R}}{1-\mathcal{R}}\right)^{c\cdot\frac{n}{2}}$$

$Z(\mathcal{R}, n)$ = Terms in the power series with $\mathcal{R}^d$ with $d = \frac{\chi+\sigma}{4} \ mod\ 4$

Agrees with, and generalizes, GKW Conjecture 1.1

The Puzzle: The naïve physical interpretation suggests we should take the constant term in the q-expansion

$$\Phi(n,\mathcal{R}) = \left[\ \nu(\tau,\mathcal{R})\ \ C(\tau,\mathcal{R})^{n^2}\ \ G(\tau,\mathcal{R})\right]_{q^0}$$

$$\Phi^J - \Phi^{J'} = \left[\nu\, C^{n^2} \Theta^{J,J'}\right]_{q^0}$$

But to get answers that agree with mathematical results we first expand in $\mathcal{R}$ and then take the constant term.

So far, we did not use any K-theory in describing the ``K-theoretic Donaldson invariants''

It would be very desirable to do so, because the 6d version, analogously formulated could be quite interesting:

Conjecture:

Integrals in elliptic cohomology of distinguished classes defined by the susy sigma model with target space $\mathcal{M}_k$ define smooth invariants of four-manifolds

SUMMARY

Part 1: We gave a necessary and sufficient condition for T-invariance of CSW with torus gauge group, and conjectured a general condition for all CSW theories.

Led to a question about whether the condition can be rephrased in terms of some (nonstandard) Witt group of fusion categories.

Part 2: SYM & Four-manifold invariants. Three questions:

Topological data for twisting of the general d=4 N=2 theory?

Invertible theory governing orientation of nonabelian SW moduli

Puzzle regarding physical derivation of K-theoretic Donaldson invariants