LINE OPERATORS IN $\mathcal{N}=2$ THEORIES

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SIMONS CENTER WORKSHOP

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WORK IN PROGRESS WITH

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$1003.5657 \leq \; ? \; \leq 2003.5657$

I. MOTIVATION & SUMMARY

- WE HAVE SEEN IN ANDY'S TALK THAT S^1- COMPACTIFICATION OF $d=4, N=2$ THEORIES PRODUCES INTERESTING FUNCTIONS $\{Y_\gamma\}$ WHICH SATISFY A TBA-LIKE EQUATION AND CAN BE USED TO CONSTRUCT HK MANIFOLDS.

- MOREOVER, FOR $T_{g,n}[A_1]$ THEORIES THESE FUNCTIONS CAN BE CONSTRUCTED FROM FOCK-GONCHAROV COORDINATES ON MODULI OF FLAT $SL(2,\mathbb{C})$ CONNECTIONS

- WE WANTED TO FIND A DIRECT PHYSICAL INTERPRETATION OF $\{y_\gamma\}$ — AND THE ANSWER IS THAT THEY ARE VEV'S OF IR LINE OPERATORS.

- WE ARE AIMING FOR A FORMULA LIKE

$$\langle L \rangle = \sum_{\gamma} \underline{\overline{\Omega}}(L,\gamma)\, y_\gamma$$

UV LINE OP

E-M CHARGES

EXPANSION COEFF'S. TURN OUT TO BE INDICES $\in \mathbb{Z}$

- WHILE UNRAVELING THIS WE FOUND SOME OTHER RESULTS OF INDEPENDENT INTEREST:

1. $\langle L \rangle$ CAN BE VIEWED AS A "CLASSICAL LIMIT" OF

$$F(L) = \sum_{\gamma} \underline{\overline{\Omega}}(L, \gamma; y) \, X_\gamma$$

$$X_\gamma X_{\gamma'} = y^{\langle \gamma, \gamma' \rangle} X_{\gamma+\gamma'}$$

$\underline{\overline{\Omega}}(L, \gamma; y) =$ CHARACTER OF A REP. OF $SU(2)$

$$= \sum_{n \geq 0} a_n [n]_y \qquad a_n \in \mathbb{Z}_+$$

2. $\Longrightarrow$ NEW PHYSICAL PROOF OF THE "MOTIVIC KSWCF"

3. THERE ARE CONCRETE ALGORITHMS FOR COMPUTING $\langle L \rangle$ & $F(L)$ IN THE $T_{g,n}[A_1]$ THEORIES

4. THE $F(L)$ ARE RELATED TO QUANTUM GEODESIC OPERATORS IN TEICHMULLER THEORY

OUTLINE

II. SUSY LINE OPERATORS

III. FRAMED BPS STATES

IV. HALOS

V. WALL-CROSSING

VI. DEFORMED PRODUCT

VII. FORMAL LINE OPERATORS

VIII. DARBOUX COORDINATE EXPANSION

IX. COMPUTING $\langle L_\zeta \rangle$ IN
A_1- THEORIES

X. SUMMARY & OPEN PROBLEMS

II. SUSY LINE OPERATORS: $d=4, \mathcal{N}=2$

- S = RG FIXED POINT WITH $SU(2,2|2)$ SYMMETRY

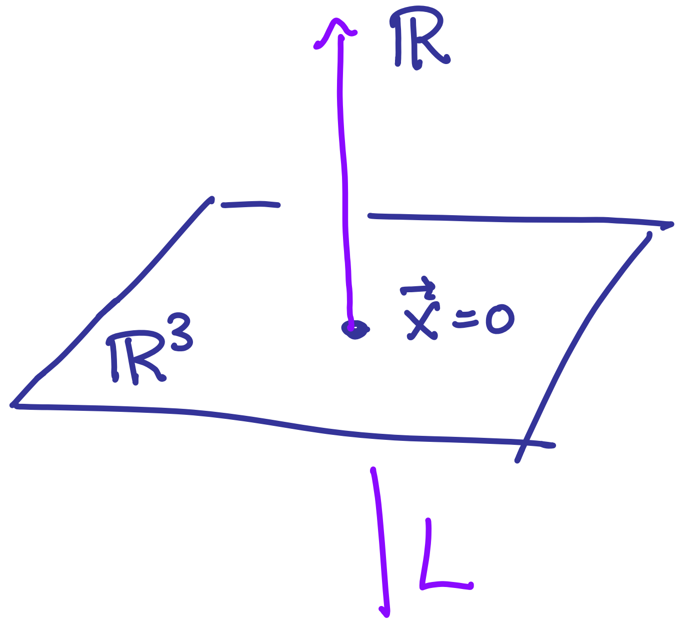
- LINE OPERATOR = BOUNDARY COND. ON S ON CONFORMAL NBD. $AdS_2 \times S^2$ [KAPUSTIN]

- L IS AN "OPERATOR" IN THE SENSE THAT WE CAN INSERT IN PATH INTEGRAL

- IN THE PRESENCE OF L THERE IS A HILBERT SPACE $\mathcal{H}_L$.

- ACTUALLY L IS AN OBJECT IN A MONOIDAL CATEGORY

SUBCATEGORY WE CONSIDER



- WE WANT TO RESTRICT ATTENTION TO A CLASS OF LINE OP'S PRESERVING 4 SUSY'S

- LET $\zeta \in U(1)$ & DEFINE A SUPERALGEBRA

$$\mathfrak{osp}(4^*|2)_\zeta \subset \mathfrak{su}(2,2|2)$$

$\boxed{\text{osp}(4^*|2)_\zeta := \text{FIXED SUBALGEBRA OF AN INVOLUTION COMBINING } \mathbb{P} \text{ WITH } U(1)_R \text{ ROTATION BY } \zeta}$

$$\text{osp}(4^*|2)_\zeta^{\text{even}} = sl(2,\mathbb{R}) \oplus so(3) \oplus su(2)$$

$\text{osp}(4^*|2)_\zeta^{\text{odd}}$ has Poincaré susy :

$$\boxed{\mathcal{R}_\alpha^{\ A} \sim Q_\alpha^{\ A} + \zeta\, \sigma^0_{\alpha\dot\beta}\, \bar{Q}^{\dot\beta A}}$$

- A LINE OP. PRESERVING THIS SYMMETRY WILL BE DENOTED $L_\zeta(\cdots)$
- WE WILL CONTINUE TO $\zeta \in \mathbb{C}^*$

EXAMPLES

1.) $G =$ CPT, SIMPLE LIE GROUP
$R =$ REP

$$L_S(G,R) = \rho_R\left(P\exp\int_{\mathbb{R}\times\vec{0}}\left(\frac{\varphi}{2S} - iA - \frac{S}{2}\bar{\varphi}\right)\right)$$

2.) COCHARACTER $v \in \mathrm{Hom}(U(1), T)$
DEFINES 't Hooft OPERATOR

3.) WILSON-'t HOOFT LABELED
BY CERTAIN COLLECTIONS (KAPUSTIN)

$$\mathcal{L}/W$$

$$\underset{\substack{\uparrow \\ \text{WILSON}}}{}\Lambda^*_{wt} \times \Lambda_{rt} \subset \mathcal{L} \subset \underset{\substack{\uparrow \\ \text{'t Hooft}}}{\Lambda^*_{rt}} \times \underset{\substack{\uparrow \\ \text{WILSON}}}{\Lambda_{wt}}$$

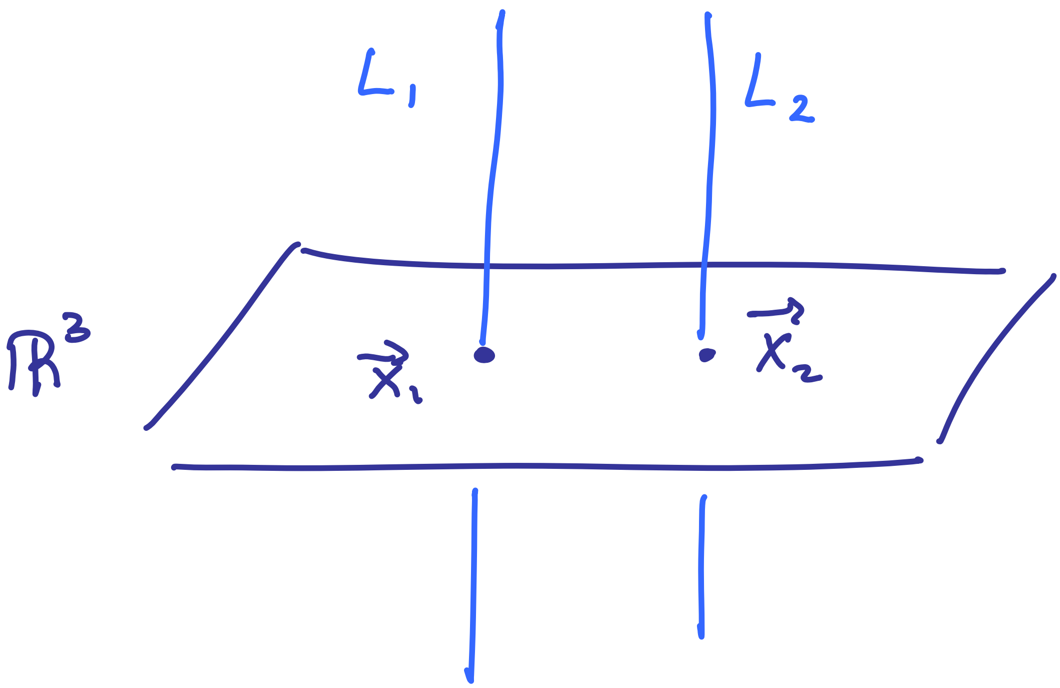
SUM

- WE CAN DEFINE THE SUM OF LINE OP'S BY SUMMING PATH INTEGRALS

- $$\mathcal{H}_{L+L'} := \mathcal{H}_{L} + \mathcal{H}_{L'}$$

DEF: L IS SIMPLE IF $L \neq L' + L''$ FOR TWO NONTRIVIAL L', L''

WE CAN DEFINE THE <u>PRODUCT</u> OF LINE OP'S BY DOUBLE INSERTION



- $\langle L_1 L_2(\vec{x}) \rangle := \lim\limits_{\substack{\vec{x}_1 \to \vec{x} \\ \vec{x}_2 \to \vec{x}}} \langle L_1(\vec{x}_1)\, L_2(\vec{x}_2) \rangle$

- COMMUTATIVE (TO BE DEFORMED)

c.f. KAPUSTIN WITTEN & KAPUSTIN SAULINA $\mathcal{N}=4$

MUTUAL LOCALITY

<u>DEF:</u> LET $U(1)_{\vec{x}_1 \vec{x}_2} \subset SU(2)_{ROT.}$

IF $\mathcal{H}_{L_1(\vec{x}_1)\, L_2(\vec{x}_2)}$ IS A REP

OF $U(1)_{\vec{x}_1 \vec{x}_2}$ (AND NOT JUST $\widetilde{U(1)}_{\vec{x}_1 \vec{x}_2}$)

THEN L_1 & L_2 ARE MUTUALLY LOCAL.

<u>EXAMPLE</u> G = SPLE GROUP

$$\Lambda_{cr} \times \Lambda_r \subset \mathcal{L} \subset \Lambda_{mw} \times \Lambda_{wt}$$

$\mathcal{L}$ = SET OF SIMPLE MUTUALLY LOCAL WILSON-'t HOOFT LOOPS IF

$$\langle P_1, Q_2 \rangle - \langle P_2, Q_1 \rangle \in \mathbb{Z}$$

$\forall\ (P_1, Q_1)$ & $(P_2, Q_2) \in \mathcal{L}$.

III. FRAMED BPS STATES & PSC

$N=2, d=4$ THEORY HAS A LOCAL SYSTEM DEFINED BY CHARGE LATTICE

$$\begin{array}{ccc} \Gamma & \longrightarrow & \mathcal{B} = \text{VM MODULI} \\ \cup\!\!\!\!| & & \cup\!\!\!\!| \\ \gamma & & u \end{array}$$

IF L IS SIMPLE THEN

$$\mathcal{H}_L = \bigoplus_{\gamma \in \Gamma_L} \mathcal{H}_{L,\gamma}$$

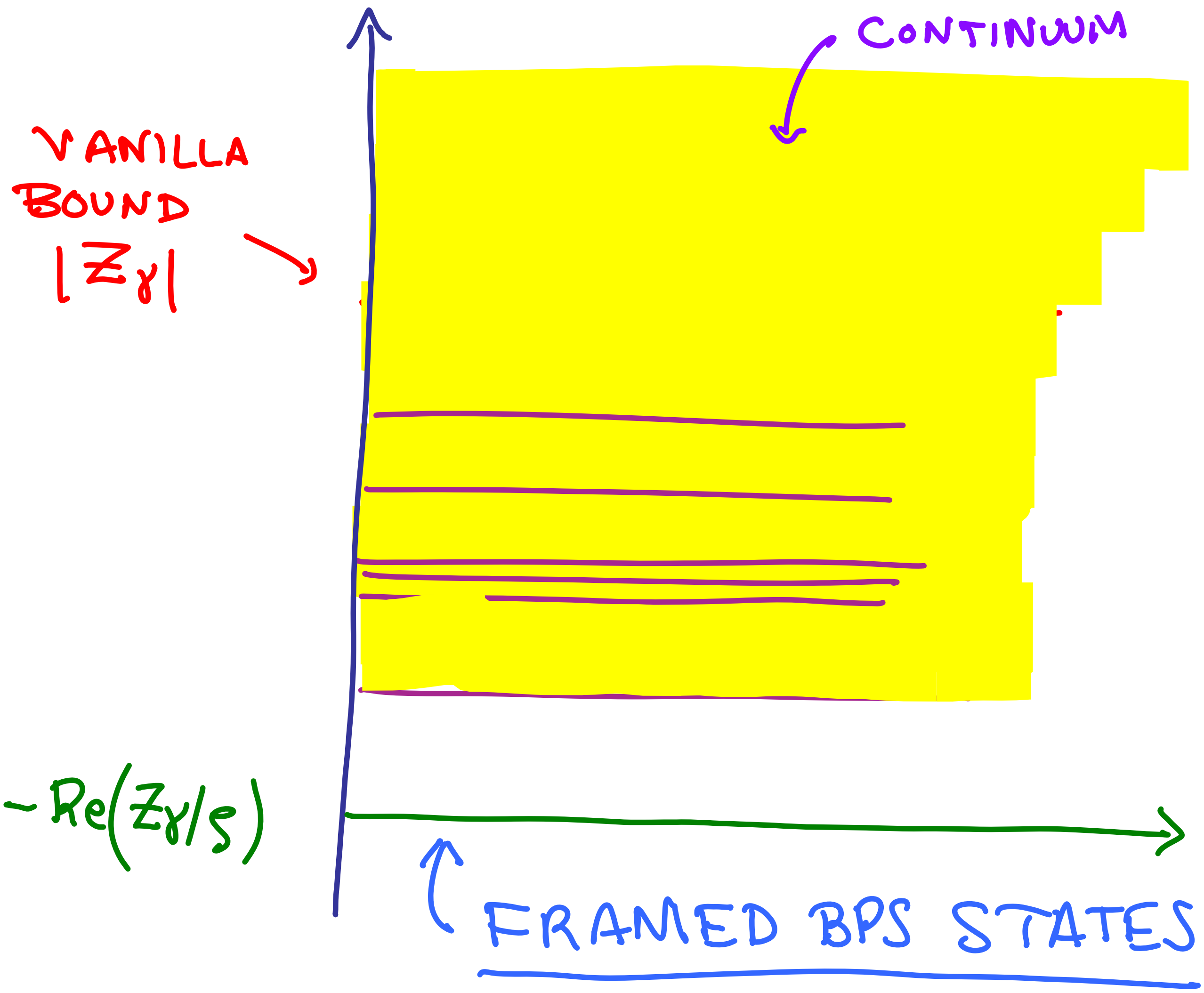
$\Gamma_L = \Gamma + \gamma_L$ TORSOR FOR Γ

EXAMPLE: EVEN IN PURE SU(2) WE CAN INTRODUCE WILSON LINE IN THE FUNDAMENTAL.

NEW BPS BOUND

$$\{ R_\alpha{}^A , R_\beta{}^B \} = 4\left(E + \mathrm{Re}(Z_\gamma/\zeta)\right) \epsilon_{\alpha\beta}\epsilon^{AB}$$

$$\Longrightarrow \quad E \geq -\mathrm{Re}(Z_\gamma/\zeta) \text{ on } \mathcal{H}_{L_{\zeta},\gamma}$$



THE PROTECTED SPIN CHARACTER

$\mathcal{H}^{BPS}_{L_\zeta,\gamma}$ IS A REP OF $so(3)_{sp} \oplus su(2)_R$

$$\boxed{\underline{\overline{\Omega}}(L_\zeta,\gamma;y) := \mathrm{Tr}_{\mathcal{H}^{BPS}_{L_\zeta,\gamma}}\, y^{2J_3}(-y)^{2I_3}}$$

- VANISHES ON MASSIVE REP'S.

- PSC FOR VANILLA BPS STATES

$$\mathcal{H}^{BPS}_{\gamma} = \left[\left(\tfrac{1}{2};0\right) \oplus \left(0;\tfrac{1}{2}\right)\right] \otimes \mathcal{H}'^{BPS}_{\gamma}$$

$$\boxed{\Omega(\gamma;y) = \mathrm{Tr}_{\mathcal{H}'^{BPS}_{\gamma}}\, y^{2J_3}(-y)^{2I_3}}$$

- THE PSC (SUGGESTED BY J.M.) IS AN IMPROVEMENT ON THE SPIN CHARACTER, WHICH IS <u>NOT</u> APRIORI AN INDEX.

- HOWEVER EXAMPLES (DIACONESCU & M., DIMOFTE & GUKOV, GMN) THUS FAR EXAMINED SUGGEST THE

STRONG POSITIVITY CONJECTURE:

FOR $d=4$, $\mathcal{N}=2$ <u>FIELD THEORY</u>

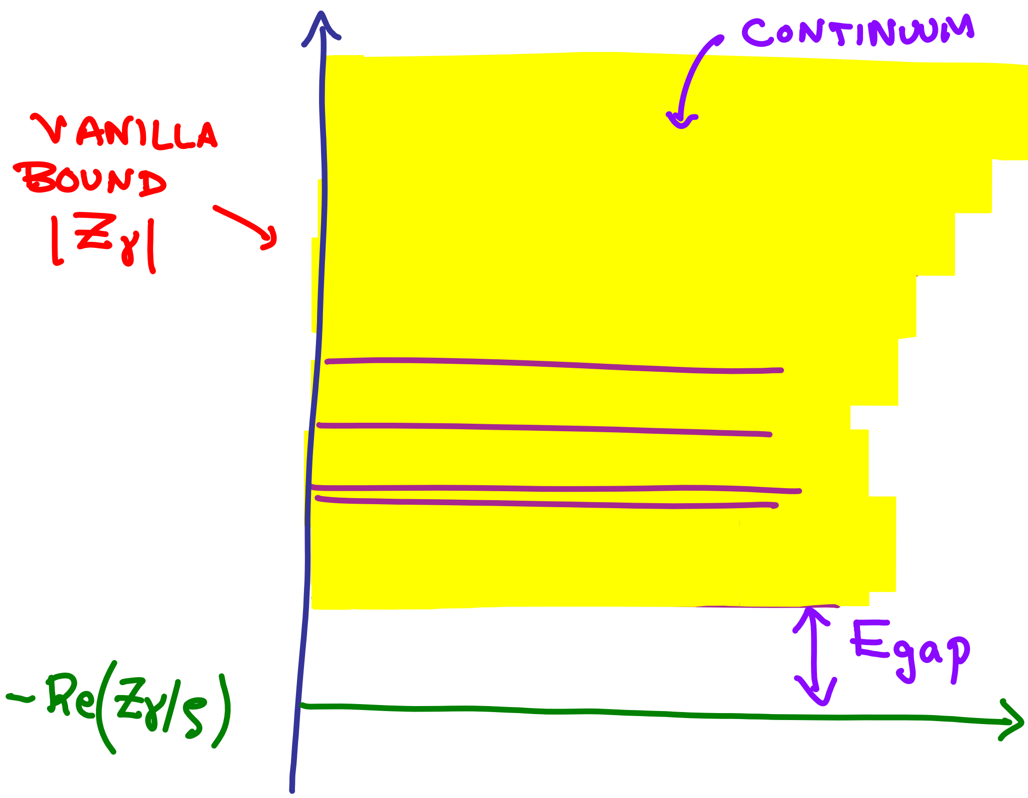
PSC = SPIN CHARACTER

($\Rightarrow$ NONTRIVIAL STATEMENTS ABOUT $\not{D}$ ON MONOPOLE MODULI SPACES...)

IV. HALOS & HALO FOCK SPACES

HOW DOES THE PSC DEPEND ON $u \in \mathcal{B}$ AND $\zeta \in U(1)$?

IT IS PIECEWISE CONSTANT BUT CAN JUMP WHEN $E_{gap} \to 0$



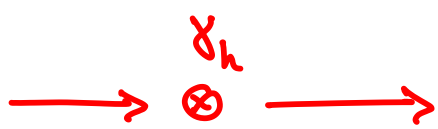
IR MODEL FOR THE BOUND STATES

STATES IN $\mathcal{H}_{L,\gamma}$:

1.) ∞-HEAVY BPS DYON OF CHARGE γ

$$E = -\mathrm{Re}\left(Z_\gamma/\zeta\right)$$

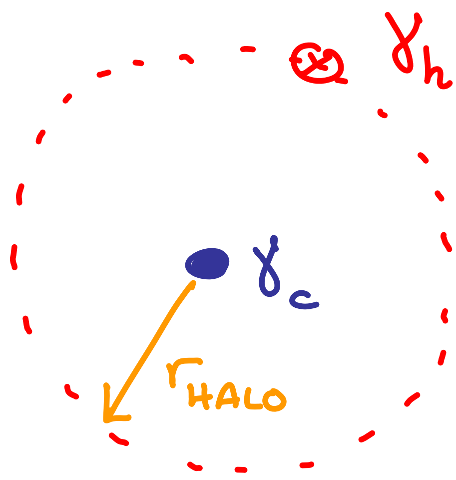
2.) $\gamma = \gamma_c + \gamma_h$



γ_c HEAVY BPS DYON OF CHARGE γ_c

$$E = -\mathrm{Re}\left(Z_{\gamma_c}/\zeta\right) + |Z_{\gamma_h}| + \tfrac{1}{2}|Z_{\gamma_h}|\, v^2 + \cdots$$

3.)



A FRAMED BPS STATE

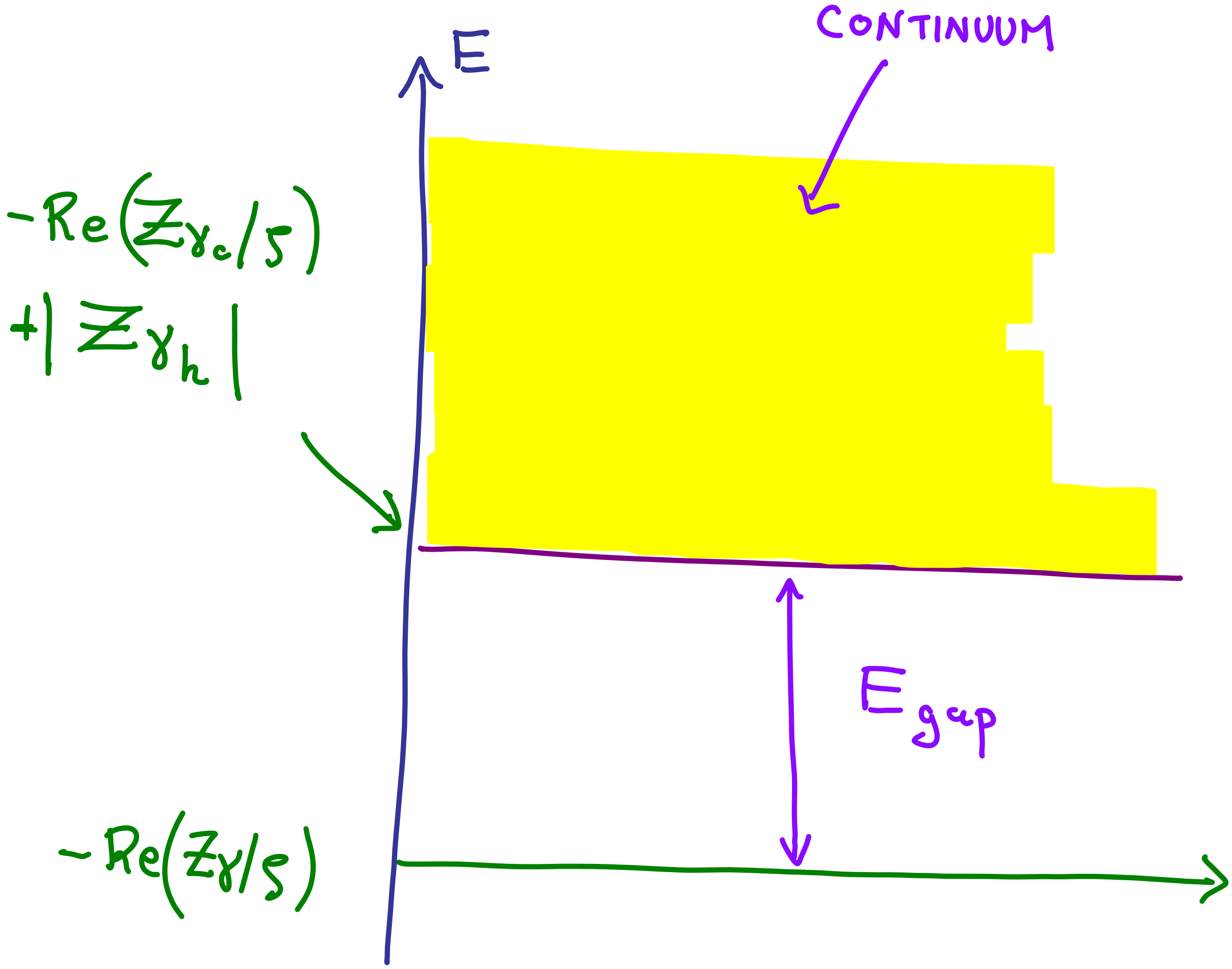
FROM $\int m\, ds + \langle \gamma_h, \mathcal{A} \rangle$ GET BOUNDSTATE

$$\boxed{r_{HALO} = \frac{\langle \gamma_h, \gamma_c \rangle}{2\,\mathrm{Im}\left(Z_{\gamma_h}(u)/\zeta\right)}}$$

$$E = -\mathrm{Re}\left(Z_{\gamma_c}/\zeta\right) - \mathrm{Re}\left(Z_{\gamma_h}/\zeta\right) = -\mathrm{Re}\left(Z_{\gamma}/\zeta\right)$$

N.B. THIS DESCRIPTION OF FRAMED BPS STATES IS ONLY VALID FOR

$$r_{HALO} \gg 1$$



$$E_{gap} = |Z_{\gamma_h}| + \mathrm{Re}(Z_{\gamma_h}/\zeta)$$

GAP CLOSES ACROSS WALLS

$$\hat{W}(\gamma_h) := \{(u,\zeta) \mid Z_{\gamma_h}(u)/\zeta < 0\}$$
$$\subset \mathcal{B} \times \mathbb{C}^*$$

LOSING BOUNDSTATES

NOTE THAT ACROSS WALLS

$$\hat{W}(\gamma_h) := \left\{ (u,\zeta) \,\middle|\, Z_{\gamma_h}(u)/\zeta < 0 \right\}$$

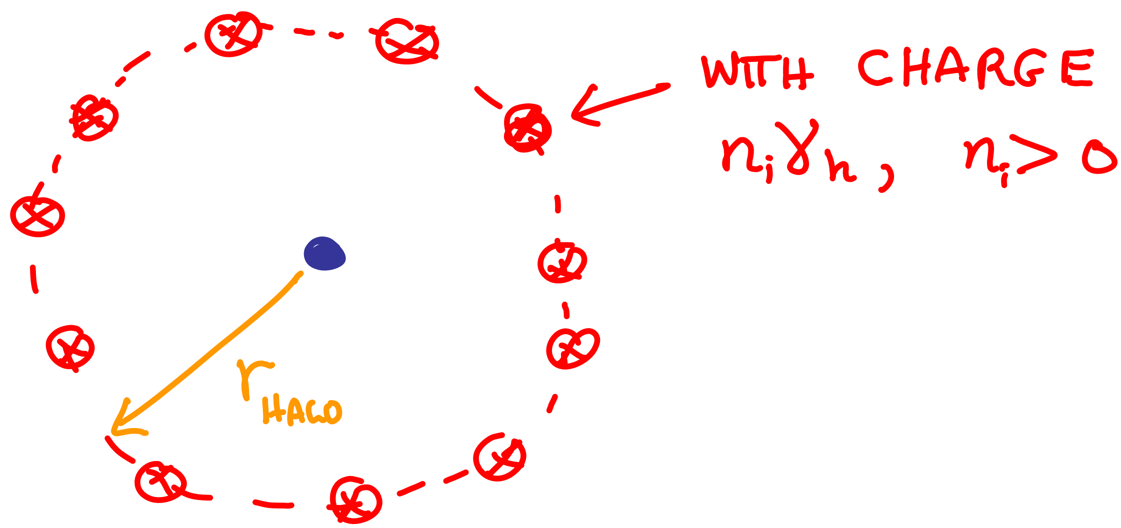
$$\subset \mathcal{B} \times \mathbb{C}^*$$

$$r_{HALO} \longrightarrow \infty$$

$$r_{HALO} = \frac{\langle \gamma_h, \gamma_c \rangle}{2\, \mathrm{Im}\left(Z_{\gamma_h}(u)/\zeta \right)}$$

HALO FOCK SPACES - PART I

SINCE $n\gamma_h$ - PARTICLES ARE MUTUALLY BPS:



MODELS A FRAMED BPS STATE FOR

$$\gamma = \left(\sum_i n_i\right)\gamma_h + \gamma_c$$

STATES BUILD UP A FOCK SPACE:

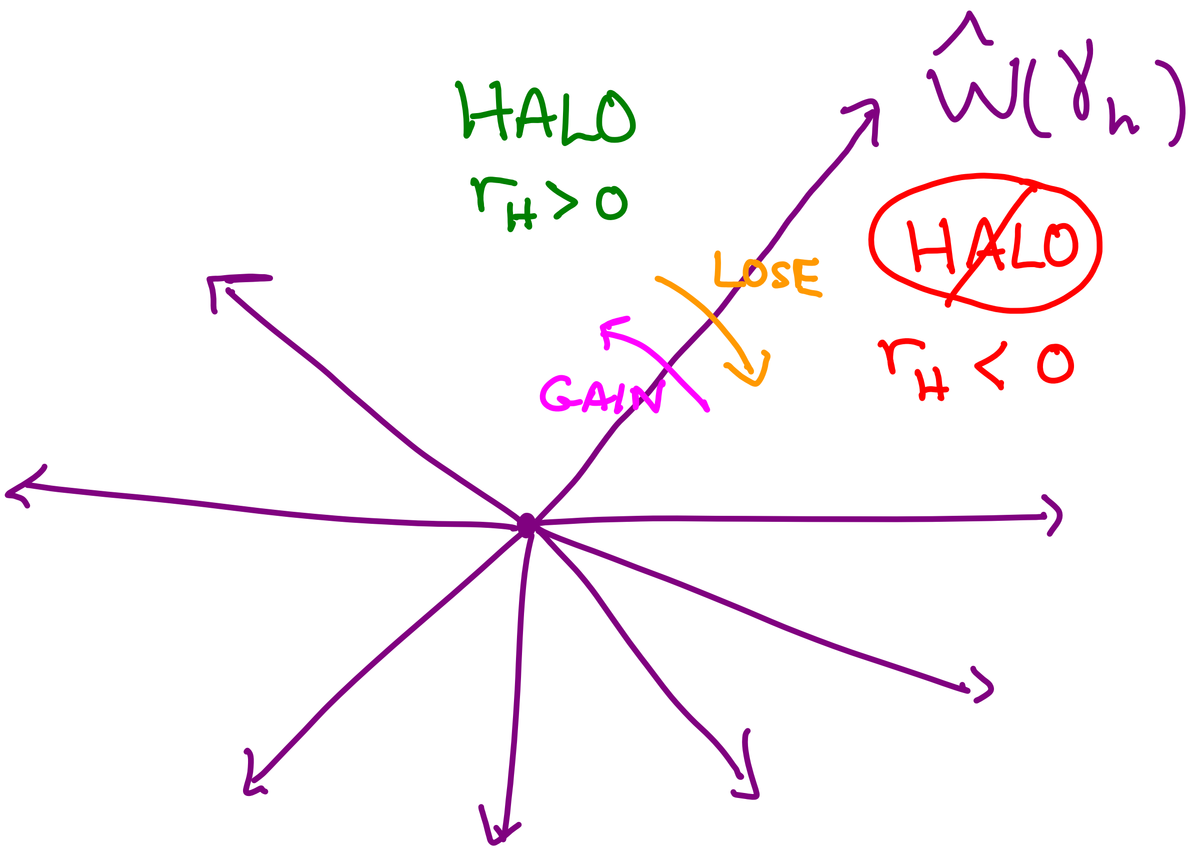
WE CAN "ADD" PARTICLES WITH CHARGE PARALLEL TO γ_h

STABILITY CONDITION FOR FOCK SPACES

$$\boxed{\langle \gamma_h, \gamma_c \rangle \, \mathrm{Im}\left(Z_{\gamma_h}(u)/\zeta \right) > 0}$$

WALL-CROSSING OF THE PSC OCCURS WHEN WE GAIN/LOSE THESE FOCK SPACES:

$$r_{HALO} \longrightarrow \infty$$



HALO FOCK SPACE - PART II

"ADDING PARTICLES" : CREATION OP'S

$$V_{\gamma_c,\gamma_h} = (J_{\gamma_c,\gamma_h}) \otimes \mathcal{H}'^{BPS}_{\gamma_h}$$

(J_{γ_c,γ_h}) = $so(3)$ IRREP, DIM $= |\langle \gamma_c, \gamma_h \rangle|$

$$\mathcal{F}^{HALO} = \mathcal{H}_{L,\gamma_c} \otimes \mathcal{F}[\underbrace{V_{\gamma_c,\gamma_h}}_{\substack{\mathbb{Z}_2\text{-GRADED} \\ \text{SPACE OF CREATION OPS}}}]$$

HYPERMULT'S = FERMIONS

VECTORMULT'S = BOSONS

$$\mathrm{Tr}_{\mathcal{H}'^{BPS}_{\gamma_h}} (-y)^{2J_3} y^{2I_3} = \sum_{-M_h}^{M_h} a_{m,\gamma_h} y^m$$

TAKING INTO ACCOUNT PARTICLES WITH CHARGE PARALLEL TO γ_h:

$$\mathcal{F}^{HALO} = \mathcal{H}_{L,\gamma_c} \otimes \bigotimes_{\ell=1}^{\infty} \mathcal{F}\left[V_{\gamma_c, \ell\gamma_h} \right]$$

V. WALL-CROSSING FORMULA

(CLASSICAL) GENERATING FUNCTION

$$F^{cl}(L) := \sum_{\gamma} \underline{\Omega}(L;\gamma;y)\, X_\gamma$$

IF $\quad X_\gamma X_{\gamma'} = X_{\gamma+\gamma'} \quad$ THEN

GAIN/LOSS OF HALOS IS EXPRESSED BY

$$X_{\gamma_c} \longrightarrow X_{\gamma_c} \prod \left(1+(-1)^m y^{m+m'} X_{\gamma_h}\right)^{a_{m,\gamma_h}}$$

$$-2J_{\gamma_c,\gamma_h} \leq m' \leq 2J_{\gamma_c,\gamma_h}$$

$$-M_h \leq m \leq M_h$$

$$\mathrm{Tr}_{\mathcal{H}'^{BPS}_{\gamma_h}} (-y)^{2J_3}\, y^{2I_3} = \sum_{-M_h}^{M_h} a_{m,\gamma_h}\, y^m$$

THIS FORMULA IS AWKWARD.

IT TURNS OUT THERE IS A BETTER WAY:

INTRODUCE QUANTUM X_γ:

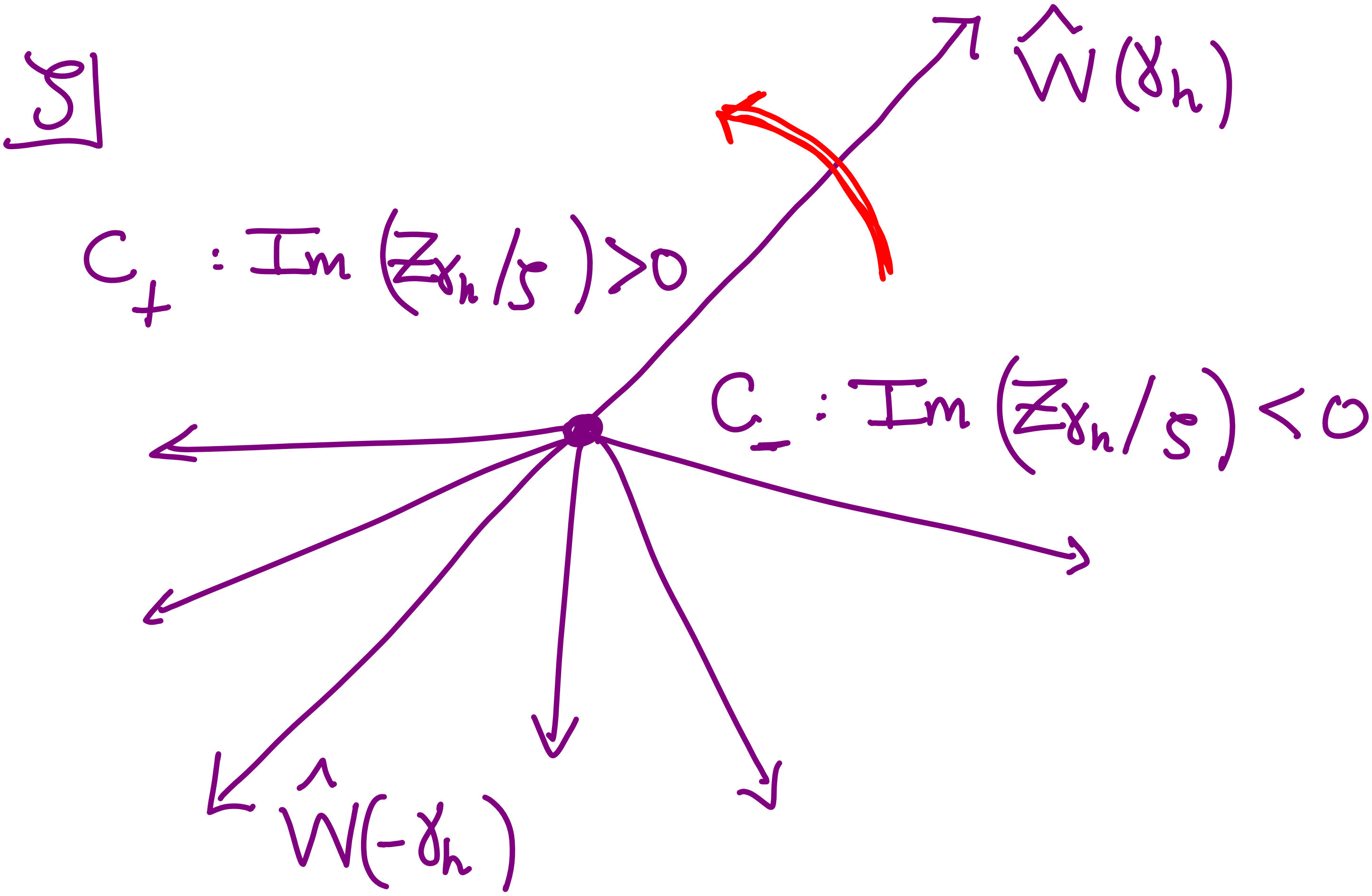
$$X_\gamma X_{\gamma'} = y^{\langle \gamma, \gamma' \rangle} X_{\gamma+\gamma'}$$

$$F(L) = \sum_\gamma \underline{\overline{\Omega}}(L;\gamma;y) X_\gamma$$

$\hat{W}(\gamma_h)$ FOR $\Omega(\gamma_h;u) \neq 0$

DIVIDE $\mathcal{B} \times \mathbb{C}^*$ INTO CHAMBERS

$F(L;c)$ = VALUE OF $F(L)$ IN CHAMBER c



$$\boxed{F(L, c^+) = S_{\gamma_h} F(L, c^-) S_{\gamma_h}^{-1}}$$

$$S_{\gamma_h} = \prod_{-M_h}^{M_h} \Phi\left((-1)^m y^m X_{\gamma_h}\right)^{a_{m,\gamma_h}}$$

$$\Phi(X) = \prod_{k=1}^{\infty} \left(1 + y^{2k-1} X\right)^{-1}.$$

= QUANTUM DILOG

- DERIVE THE "MOTIVIC KSWCF":

TWO DIFFERENT PATHS BETWEEN A FIXED PAIR OF CHAMBERS MUST INDUCE THE SAME TRANSFORMATION!

- CONSISTENT WITH DIMOFTE/GUKOV/SOIB

- CLASSICAL LIMIT: $y = -1$

$$\hat{X}_\gamma \hat{X}_{\gamma'} = (-1)^{\langle\gamma,\gamma'\rangle} \hat{X}_{\gamma+\gamma'}$$

TWISTED GROUP LAW ON COMMUTATIVE VARIABLES.

$$\hat{X}_\gamma \longrightarrow K_{\gamma_h}^{-\Omega(\gamma_h)}(\hat{X}_\gamma)$$

$$K_{\gamma_h}(\hat{X}_\gamma) = \hat{X}_\gamma (1 - \hat{X}_{\gamma_h})^{\langle\gamma,\gamma_h\rangle}$$

CLUSTER TRANSFORMATION

VI. DEFORMED PRODUCT

WE CAN DEFINE A

y-PRODUCT SUCH THAT

$$\boxed{F(L_1 \circ_y L_2) = F(L_1)F(L_2)}$$

$$\left(\mathcal{H}_{L_1(\vec{x}_1)L_2(\vec{x}_2)}\right)_{\gamma_0}$$

$$= \bigoplus_{\gamma+\gamma'=\gamma_0} \mathcal{H}_{L_1,\gamma} \otimes \mathcal{H}_{L_2\gamma'} \otimes N_{\gamma,\gamma'}$$

$$\mathrm{Tr}_{N_{\gamma,\gamma'}}\, y^{2J_3} = y^{\langle\gamma,\gamma'\rangle}$$

WE HAVE NOW REVIEWED THE BASICS OF LINE OP'S.

REMAINDER OF THE TALK IS DEVOTED TO 3 RESULTS MENTIONED IN THE INTRO.

A. CONSEQUENCES OF THE STRONG POSITIVITY CONJECTURE

B. DARBOUX-COORDINATE EXPANSION

C. COMPUTATION OF $\langle L_\wp \rangle$ ON $\mathbb{R}^3 \times S^1$ FOR $T_{g,n}[A_1]$ THEORIES

VIII. FORMAL LINE OP'S

WE NOW FORMALIZE THE NOTION OF LINE OPERATOR.

THIS IS A PURELY MATHEMATICAL CONSTRUCTION.

DATA

1. LOCAL SYSTEM $\Gamma \to \mathcal{B}$
2. $\langle \cdot , \cdot \rangle : \Lambda^2 \Gamma \longrightarrow \mathbb{Z}$
3. $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$
4. $\Omega(\gamma; u; y)$ SATISFYING THE KSWCF

DIVIDE $\mathcal{B} \times \mathbb{C}^*$ INTO CHAMBERS BY WALLS $\hat{W}(\gamma)$ WITH $\Omega(\gamma;u) \neq 0$.

DEF: A STRONGLY POSITIVE FORMAL LINE OPERATOR IS A COLLECTION $F(c)$ ON CHAMBERS

- $F(c) = \sum_{\gamma} P^{c}_{\gamma} \cdot X_{\gamma}$, FINITE SUM,

$$P^{c}_{\gamma} = \text{TRUE SPIN CHARACTER}$$

$$= \sum_{n \geq 0} a_n \left(\frac{y^n - y^{-n}}{y - y^{-1}} \right)$$

$$= \sum_{n \geq 0} a_n \, [n]_y \qquad a_n \in \mathbb{Z}_+$$

- $F(c^+) = S_{\gamma_h} F(c^-) S_{\gamma_h}^{-1}$

STRONG POSITIVITY IS A NONTRIVIAL CONSTRAINT :

$$X_{\gamma_c} \longrightarrow \Phi(X_{\gamma_h}) X_{\gamma_c} \Phi(X_{\gamma_h})^{-1}$$

$$= \begin{cases} \displaystyle\sum_{j=0}^{N} ch(\Lambda^j \rho_N)\, X_{\gamma_c + j\gamma_h} & \langle \gamma_c, \gamma_h \rangle = -N < 0 \\ \displaystyle\sum_{j=0}^{\infty} (-1)^j ch(S^j \rho_N)\, X_{\gamma_c + j\gamma_h} & \langle \gamma_c, \gamma_h \rangle = +N > 0 \end{cases}$$

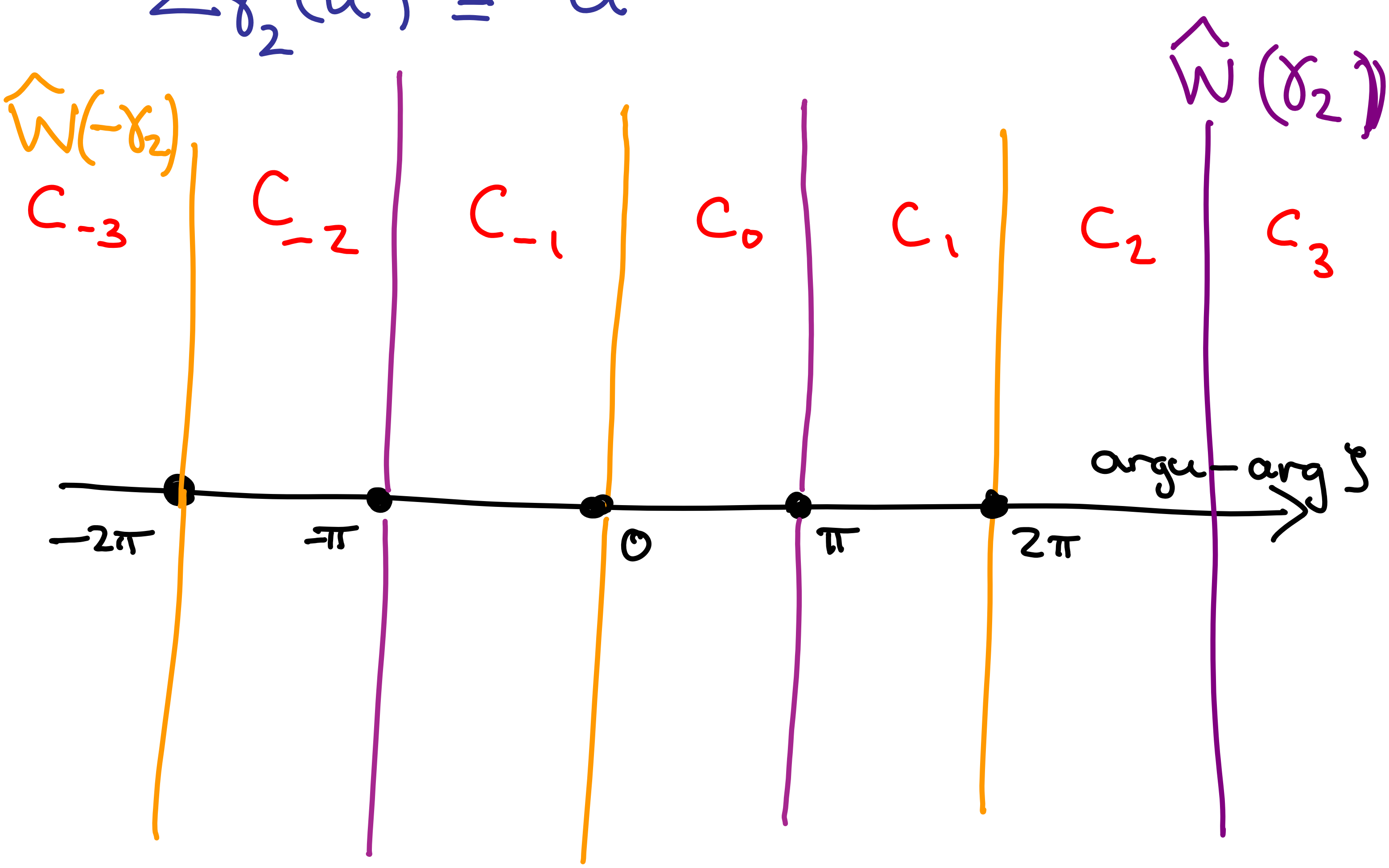
THUS YOU CAN'T JUST START WITH SOME ARBITRARY F(C) IN SOME CHAMBER AND GENERATE THE REST

EXAMPLE: U(1) THEORY WITH SINGLE HYPERMULTIPLET

$\mathcal{B} = \mathbb{C} - \{0\}$, $\Gamma = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$, $\langle \gamma_1, \gamma_2 \rangle = 1$

$$\Omega(\gamma) = \begin{cases} 1 & \gamma = \pm\gamma_2 \\ 0 & \text{ELSE} \end{cases}$$

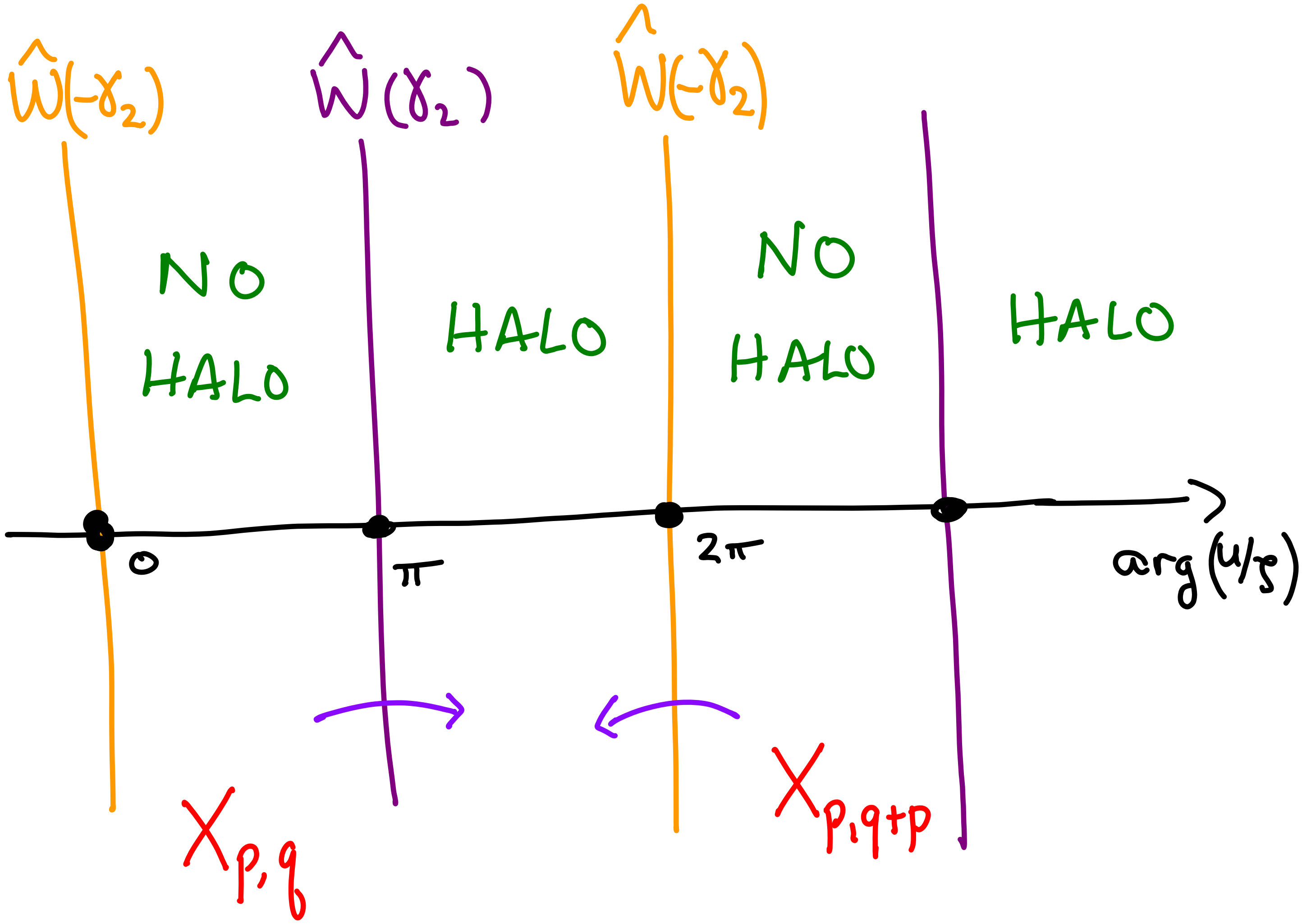
$$Z_{\gamma_2}(u) = u$$



TRY TO DEFINE $F_{p,q}$ BY

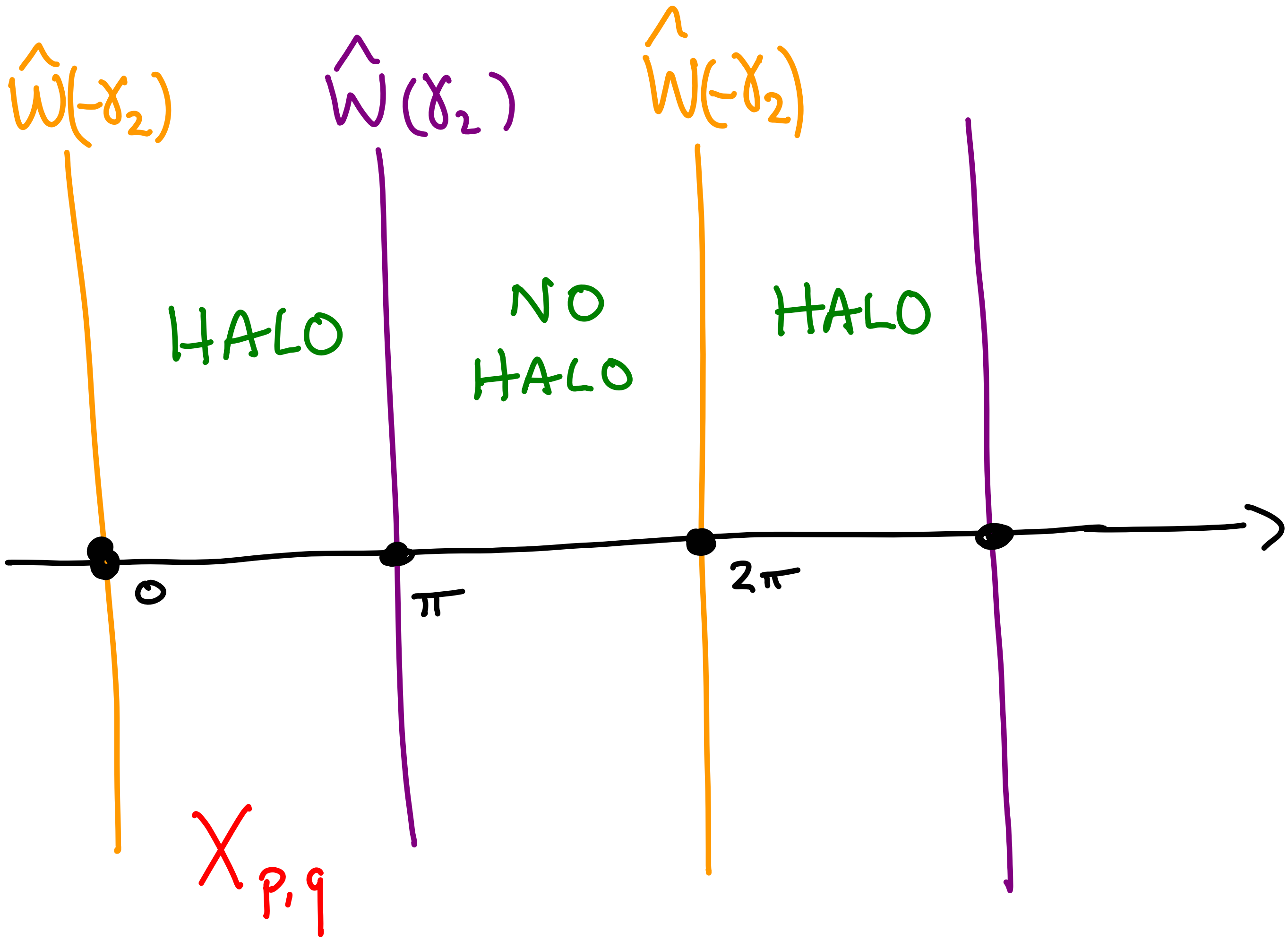
$$F_{p,q}(c_0) = X_{p,q}$$

FOR $p > 0$



$$F_{p,q}(c_1) = \sum_{j=0}^{p} \mathrm{ch}\,\Lambda^j \rho_p \cdot X_{p,q+j} = \sum_{j=0}^{p} \begin{bmatrix} p \\ j \end{bmatrix} X_{p,q+j}$$

BUT FOR $p < 0$



$$F_{p,q}(c_1) = \sum_{j=0}^{\infty} (-1)^j \, ch(S^j \rho_p) \, X_{p,q+j}$$

$$= X_{p,q} - [p] \, X_{p,q+1} \pm \cdots$$

<u>STRONG</u> <u>POSITIVITY</u> <u>FAILS</u>

WE CAN DETERMINE THE FULL ALGEBRA OF LINE OPERATORS

SIMPLE LINE OP'S:

$$F_{p,q} = \begin{cases} X_{p,q} & \text{in } C_0 \quad p \geq 0 \\ X_{p,q} & \text{in } C_1 \quad p \leq 0 \end{cases}$$

$$F_{p,q} F_{r,s} = \begin{cases} y^{ps-qr} F_{p+r,q+s} & p,r \text{ SAME SIGN} \\ y^{ps-qr} F_{p+r,q+s} + \cdots & \text{ELSE} \end{cases}$$

$$F_{1,0} F_{-1,0} = 1 + y F_{0,1}$$

SIMPLE GENERALIZATION

$$\Omega(\gamma) = \begin{cases} 1 & \gamma = \pm Q\gamma_2 \,, \quad Q > 1 \\ 0 & \text{else} \end{cases}$$

$$F_{1,0} F_{-1,0} = \sum_{j=0}^{Q} \begin{bmatrix} Q \\ j \end{bmatrix} y^{Qj} F_{0,Qj}$$

$\exists$ SIMILAR EXAMPLES FOR AD & SU(2) THEORIES...

CONJECTURE:

THE ALGEBRA OF FORMAL LINE OPERATORS IS CANONICALLY ISOMORPHIC TO THE ALGEBRA OF TRUE LINE OPERATORS

VII. DARBOUX-COORDINATE EXPANSION

ON $\mathbb{R}^3 \times S^1_R$ WE HAVE A LOOP OPERATOR AND

$\langle L_\zeta \rangle$ = FUNCTION ON MODULI OF $d=3$ VACUA $\mathcal{M} \longrightarrow \mathcal{B}$

$$\boxed{\langle L_\zeta \rangle = \sum_\gamma \overline{\underline{\Omega}}(L_\zeta, \gamma)\, \mathcal{Y}_\gamma}$$

(2: $\exists$ SEVERAL MODULI SPACES $\mathcal{M}_\mathcal{L}$ WITH ISOGENOUS FIBERS.)

"PROOF"

1. SUSY $\Rightarrow$ $\langle L_\zeta \rangle$ HOLOMORPHIC FUNCTION ON $\mathcal{M}^\zeta$

2. $R \to \infty$ ASYMPTOTICS:

$$\langle L_\zeta \rangle = \mathrm{Tr}_{\mathcal{H}_{L_\zeta}} (-1)^F e^{-2\pi R H + i\theta \cdot Q} \sigma(Q)$$

$\theta \sim$ BOUNDARY COND'S ON (ELEC, MAG) WILSON LINES $\in \mathrm{Hom}(\Gamma, \mathbb{R}/2\pi\mathbb{Z})$

$Q \sim$ CHARGE OPERATOR

$\sigma \sim$ Q.R. OF $(-1)^{\langle \gamma, \gamma' \rangle}$ DUE TO SELF-DUALITY

$\left(e^{i\hat{\theta}\cdot\gamma} = e^{i\theta\cdot\gamma} \sigma(\gamma) \right.$ IS CANONICAL, BUT TWISTED $\left. \right)$

$R \longrightarrow \infty$ PROJECTS TO FRAMED BPS STATES

$$\langle L_\zeta \rangle \sim \sum_\gamma \underline{\overline{\Omega}}(L_\zeta, \gamma)\, y_\gamma^{s.f.}$$

$$y_\gamma^{sf} = e^{\pi R \zeta^{-1} Z_\gamma + i\theta\cdot\gamma + \pi R \zeta \overline{Z}_\gamma}\, \sigma(\gamma)$$

SO WE MAKE THE ANSATZ

$$\langle L_\zeta \rangle = \sum_\gamma \underline{\overline{\Omega}}(L_\zeta, \gamma)\, \tilde{y}_\gamma$$

$\tilde{y}_\gamma$ HOLOMORPHIC WITH

$$\tilde{y}_\gamma \sim y_\gamma^{sf} \qquad R \to \infty$$

(ALSO $\zeta \to 0, \infty$)

3. WALL-CROSSING

$\overline{\underline{\Omega}}(L_\zeta, \gamma)$ HAS WALL-CROSSING:

$$F^{cl} = \sum \overline{\underline{\Omega}}(L_\zeta, \gamma)\hat{x}_\gamma$$

TRANSFORMS BY

$$\hat{x}_\gamma \longrightarrow \hat{x}_\gamma \left(1 - \hat{x}_{\gamma_h}\right)^{-\langle \gamma, \gamma_h \rangle \Omega(\gamma_h)}$$

$$= K_{\gamma_h}^{-\Omega(\gamma_h)}(\hat{x}_\gamma)$$

BUT $\langle L_\zeta \rangle$ SHOULD HAVE

NO WALL-CROSSING:

THERE IS NO PHASE TRANSITION IN THE UV THEORY.

$\Longrightarrow$ $\tilde{y}_\gamma$ TRANSFORM LIKE y_γ

$\Longrightarrow$ $\tilde{y}_\gamma = y_\gamma$ $\blacksquare$

- THUS DARBOUX COORD'S $\mathcal{Y}_\gamma$ CAN BE INTERPRETED AS "IR LINE OPERATOR VEV'S," OR CHIRAL RING OPERATORS IN THE 3D σ-MODEL

$$\mathbb{R}^3 \longrightarrow \mathcal{M}$$

- MOREOVER THE N.C. ALGEBRA OF LINE OPERATORS DEFINES A DEFORMATION OF THE ALGEBRA OF FUNCTIONS ON $\mathcal{M}$

? RELATION TO ALGEBRA $\mathcal{H}_{BB}$ OF COISOTROPIC BRANES OF KAPUSTIN-WITTEN & NEKRASOV-WITTEN?

EXAMPLE: U(1) THEORY

$\mathcal{M}$ = PERIODIC TAUB-NUT

$$U = \mathcal{Y}_{0,1} = \exp\left(\frac{\pi R}{\zeta} a + i\hat{\theta}_e + \pi R \zeta \bar{a}\right)$$

$$V_+ = \mathcal{Y}_{1,0} \qquad \mathrm{Im}\, Z_{\gamma_2}/\zeta > 0$$

$$V_- = \mathcal{Y}_{-1,0} \qquad \mathrm{Im}\, Z_{\gamma_2}/\zeta < 0$$

$$V_+ V_- = 1 - U$$

COMPARE WITH

$$F_{1,0} F_{-1,0} = 1 + y F_{0,1}$$

@ $y = -1$!!

U(1) WITH $Q > 1$

$$V_+ = y_{1,0} \qquad V_- = y_{-1,0}$$

$$V_+ V_- = \left(1 + (-U)^Q\right)^Q$$

COMPARE WITH

$$F_{1,0} F_{-1,0} = \sum_{j=0}^{Q} \begin{bmatrix} Q \\ j \end{bmatrix} y^{Qj} F_{0,Qj}$$

IX. COMPUTING $\langle L_\zeta \rangle$ IN $T_{g,n}[A_1]$ THEORIES

$T_{g,n}[\mathfrak{g}] :=$ (2,0) $\mathfrak{g}$-THEORY WITH TWIST ON $C \times \mathbb{R}^{1,3}$

$\exists$ SUSY SURFACE OPERATORS IN 6D: $\mathbb{S}(\mathcal{R}, \Sigma)$

$\mathcal{R}$: $\mathfrak{g}$ REP, Σ: SURFACE

1. ON THE COULOMB BRANCH ~

$$\sum_{\nu \in Wt(R)} \exp\left\{2\pi i \left\langle \nu, \int_{\Sigma} B + k\, n^I Y^I \mathrm{vol}(\Sigma) \right\rangle\right\}$$

2. REDUCED ON S^1

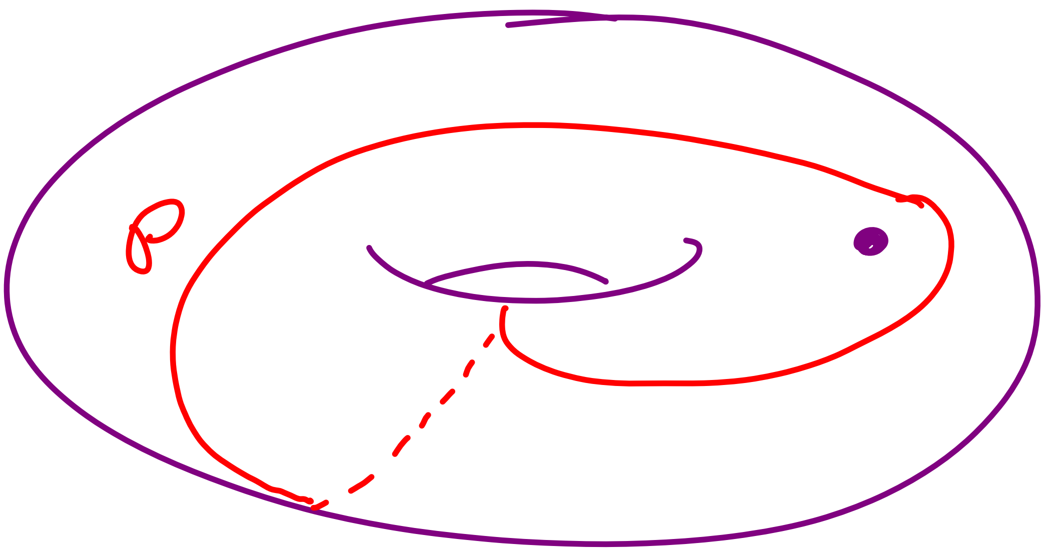
$$\mathrm{Tr}_R \, \underline{P} \exp \, 2\pi i \int_{\wp} \left(A + k\, n^I Y^I \, ds \right)$$

$\wp \subset \mathbb{R}^5$

FOR $T_{g,n}[\mathfrak{g}]$ THEORIES TAKE

$$\Sigma = \mathbb{R} \times \wp$$

$\wp$ = CLOSED, NONINTERSECTING CURVE ON C



$\mathbb{S}(\mathcal{R}, \Sigma)$ DESCENDS TO A LINE OPERATOR OF TYPE ζ

$\zeta \sim$ "ANGLE BETWEEN n^I & $\wp$"

IN THIS WAY WE CAN CONSTRUCT SIMPLE LINE OPERATORS

$$L_{\zeta}(\mathfrak{R}, \wp)$$

$\wp \sim$ ISOTOPY CLASS OF CLOSED CURVE IN C

NOW CONSIDER $T_{g,n}[\mathfrak{g}] / \mathbb{R}^3 \times S^1$

AND THE VEV'S

$$\langle L_{\zeta} \rangle = \sum \underline{\overline{\Omega}}(L_{\zeta}, \gamma)\, Y_{\gamma}$$

RECALL:

$$
\begin{tikzcd}[column sep=tiny]
 & (2,0)\ \mathfrak{g}\text{-THEORY}/\mathbb{R}^3\times S^1_R\times C \arrow[dl, "C"'] \arrow[dr, "S^1_R"] & \\
T_{g,n}[\mathfrak{g}]/\mathbb{R}^3\times S^1_R \arrow[dr, "S^1_R"'] & & 5D\ \mathfrak{g}\text{-SYM}/\mathbb{R}^3\times C \arrow[dl, "C"] \\
 & \begin{array}{c}\sigma\text{-MODEL}\\ \mathbb{R}^3 \longrightarrow \mathcal{M}\end{array} &
\end{tikzcd}
$$

$$\mathcal{M}\left(\mathbb{R}^3\times S^1_R\times C\right) = \text{HITCHIN MODULI SPACE}$$

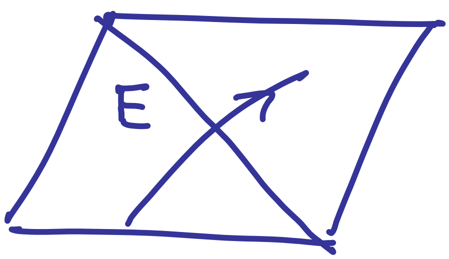
$\mathcal{M}^{\zeta}$ = MODULI OF FLAT CONNECTION

$$\mathcal{A}_\zeta = R\frac{\varphi}{\zeta} + A + R\zeta\bar{\varphi}$$

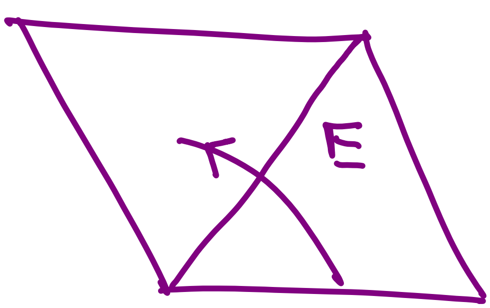
$$\begin{tikzcd}
 & \langle \mathcal{S}(R, S^1_R \times p)\rangle \arrow[dl, "C"'] \arrow[dr, "S^1"] & \\
\langle L_\zeta(R,p)\rangle & & \mathrm{Tr}_R\, P\exp\oint_p \mathcal{A}_\zeta
\end{tikzcd}$$

$$\boxed{\langle L_\zeta(R,p)\rangle = \mathrm{Tr}_R\, \mathrm{Hol}(p, \mathcal{A}_\zeta)}$$

FOR $\mathfrak{g} = A_1$ THE HOLONOMY IS COMPUTABLE IN TERMS OF FG COORD'S VIA "TRAFFIC RULES"



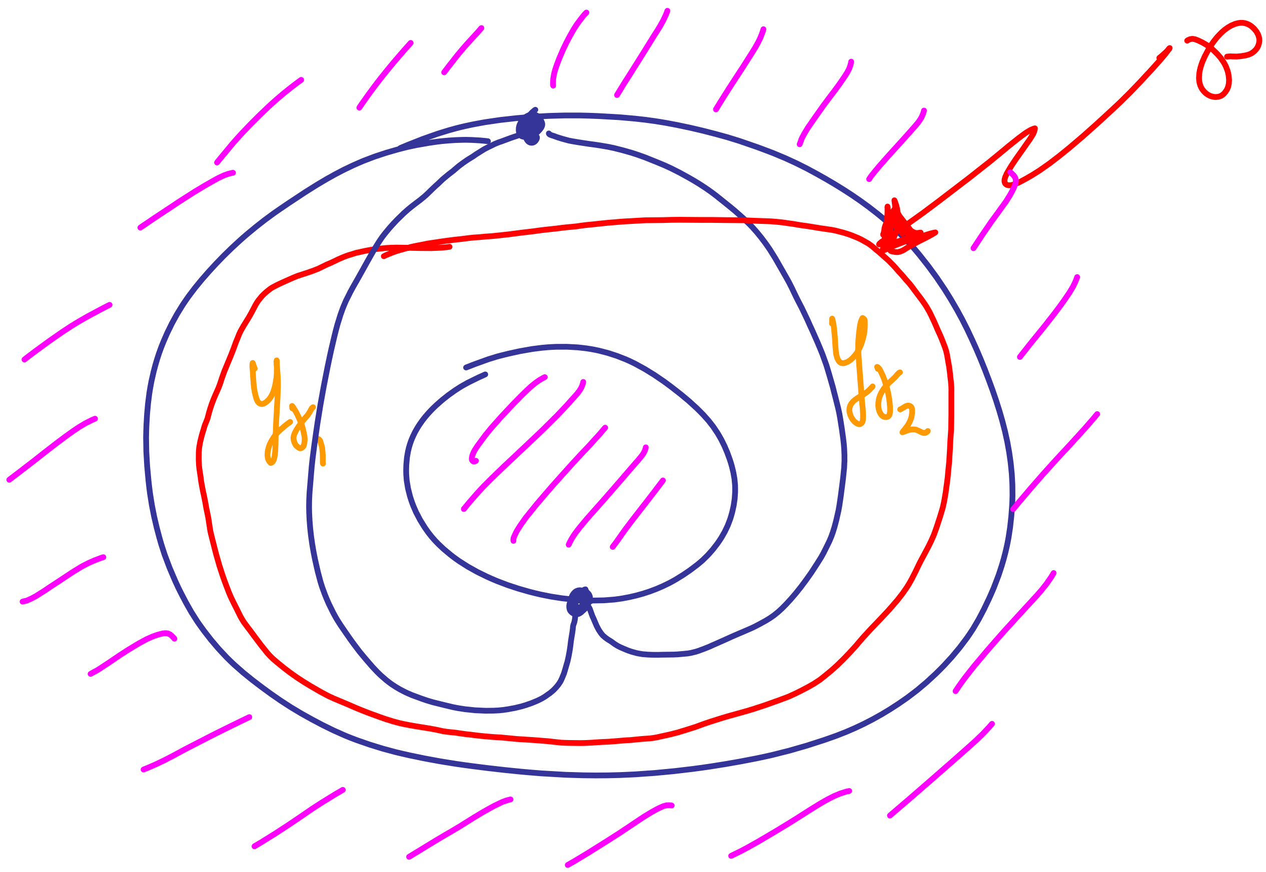
$$R(E) = \epsilon(E) \begin{pmatrix} \sqrt{y_E} & \sqrt{y_E} \\ 0 & -\frac{1}{\sqrt{y_E}} \end{pmatrix}$$



$$L(E) = \epsilon(E) \begin{pmatrix} -\sqrt{y_E} & 0 \\ \frac{1}{\sqrt{y_E}} & \frac{1}{\sqrt{y_E}} \end{pmatrix}$$

$$\epsilon(E) = \pm i$$

KEY EXAMPLE: WILSON LOOP



$$\langle L_\zeta(\underline{2},\gamma)\rangle = \underbrace{\sqrt{y_{\gamma_1}y_{\gamma_2}} + \frac{1}{\sqrt{y_{\gamma_1}y_{\gamma_2}}}}_{\text{NAIVE}} + \underbrace{\sqrt{\frac{y_{\gamma_1}}{y_{\gamma_2}}}}_{\text{SURPRISE!}}$$

QUANTUM HOLONOMY

$$\langle L_\wp(2,p)\rangle = \sqrt{y_{\gamma_1} y_{\gamma_2}} + \frac{1}{\sqrt{y_{\gamma_1} y_{\gamma_2}}} + \sqrt{\frac{y_{\gamma_1}}{y_{\gamma_2}}}$$

BY STRONG POSITIVITY THIS CAN BE PROMOTED TO A GENERATOR OF PROTECTED SPIN CHARACTERS !

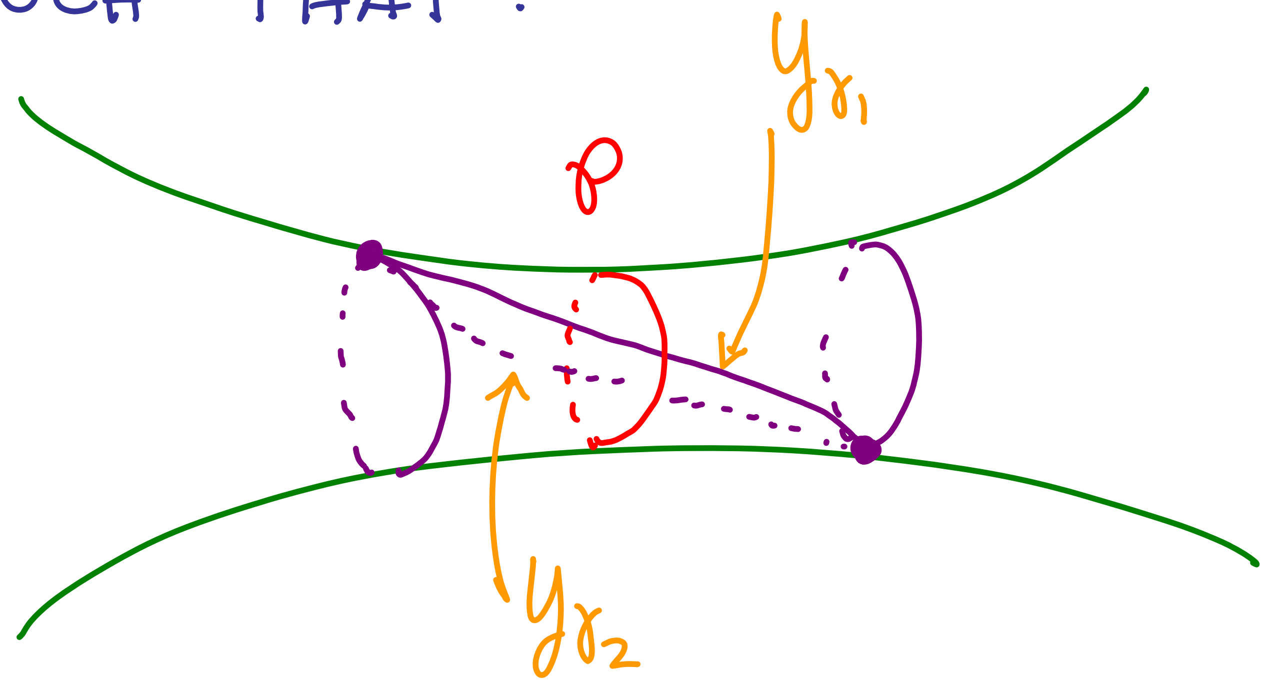
$$F(L_\wp) = X_{\frac{1}{2}(\gamma_1+\gamma_2)} + X_{-\frac{1}{2}(\gamma_1+\gamma_2)} + X_{\frac{1}{2}(\gamma_1-\gamma_2)}$$

ALGORITHM TO COMPUTE $F(L_\zeta(2,\rho))$

- u, ζ DETERMINES A WKB TRIANGULATION OF C

- NBD. OF ρ IS AN ANNULUS.

SO CHOOSE A TRIANGULATION OF C SUCH THAT:



- THEN FLIP USING THE QUANTUM DILOG.

REMARK: THIS ALGORITHM APPEARS TO COINCIDE WITH THE PRESCRIPTION BY TESCHNER FOR COMPUTING QUANTUM GEODESIC LENGTHS IN FOCK COORDINATES IN THE QUANTIZATION OF TEICHMULLER SPACE.

X. SUMMARY & OPEN PROBLEMS

S1 WE DEFINED LINE OP'S L_ζ AND THEIR PROTECTED SPIN CHAR'S. THEY SATISFY THE KSWCF AND DEFINE A DEFORMATION OF THE ALGEBRA OF FUNCTIONS ON $\mathcal{M}$.

S2 WE DEFINED STRONGLY POSITIVE FORMAL LINE OPERATORS – CONJECTURALLY ISOMORPHIC TO LINE OP'S OF $\mathcal{N}=2$.

S3 WE DESCRIBED CONCRETE ALGORITHMS FOR COMPUTING THE FRAMED BPS DEGEN'S & PSC'S IN $T_{g,n}[A_1]$ – THEORIES

O1. CLARIFY RELATION TO QUANTIZATION OF TEICHMULLER AND HITCHIN MODULI

O2: CLARIFY RELATION TO "CANONICAL BASES" AND "TOTAL POSITIVITY" (LUSZTIG, FOMIN-ZELEVINSKY, ...)

O3: HOW TO COMPUTE AT HIGHER RANK?

O4: QUANTUM DILOG ALSO APPEARS IN NONCOMPACT CSW-THEORY

O5: EXTENSION TO SUGRA

O6: GEOMETRY OF MONOPOLE MODULI SPACE

O7: APPLICATIONS TO SURFACE OP'S AND 2D/4D W.C.F.