SCGP March 21, 2019

This talk $\approx$ Feb. 6, 2019

1. MOTIVATION
2. RECALL W.C.F.
3. REVIEW & REFINE CVWCF
4. BRANIFY
5. MULTI-VALUED W

8:49 ① : Motivation

WCF for BPS $\Omega(\gamma) \sim \mathrm{Tr}_{\mathcal{H}^{BPS}} (-1)^F$

"Categorify" : $\mathcal{H}^{BPS}$?

old stuff

Gaiotto + Witten c. 2015

Dimofte + Gaiotto c. 2016 ... w/ AHSAN KHAN

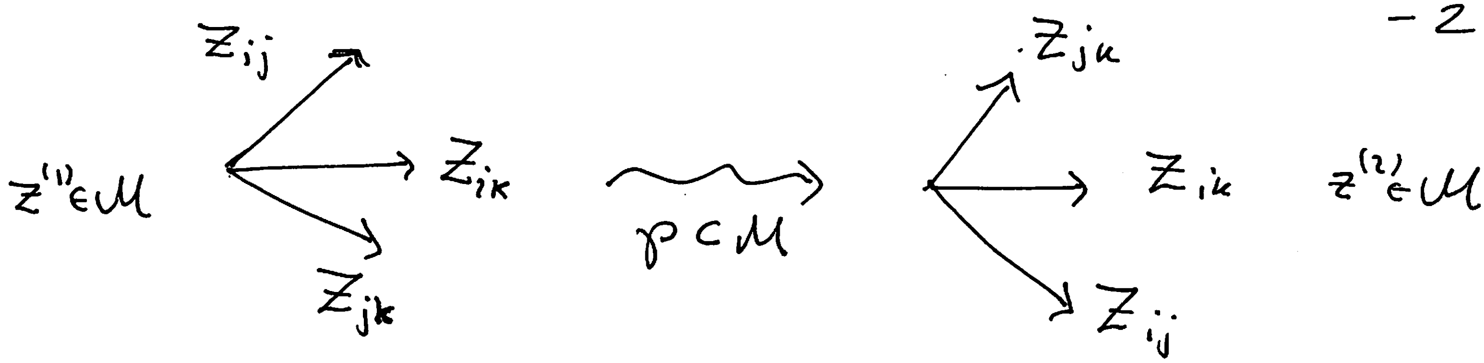
② WCF :

A.) CVWCF (1992) : $\{ 1+1 \text{ QFT } (2,2) \} = \mathcal{M} \ni z$

Vacua $i, j, k \dots$ N of them

$i \neq j$ BPS states i ——— j $Z_{ij}^{(z)} \in \mathbb{C}$

$\mu_{ij} \sim \mathrm{Tr}_{\mathcal{H}_{ij}^{BPS}} (-1)^F$ piecewise cont. CFIV.



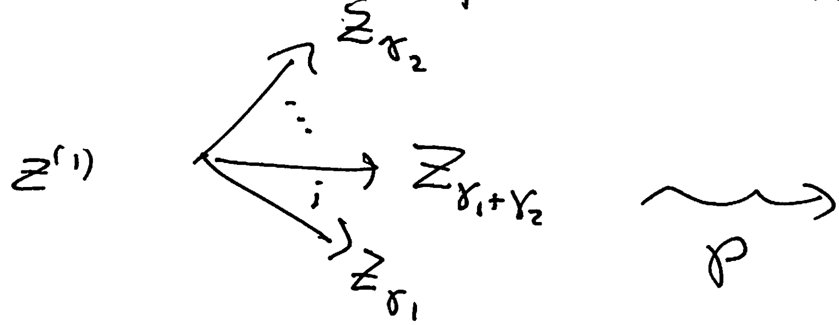
$$S_{ij} = \mathbb{1}_N + \mu_{ij} e_{ij} = \left(\mathbb{1}_N + e_{ij}\right)^{\mu_{ij}}$$

$$\left(S_{ij} S_{ik} S_{jk}\right)^{(1)} = \left(S_{jk} S_{ik} S_{ij}\right)^{(2)}$$

B.) KSWCF (2008)

$$\begin{array}{c} \Gamma \\ \downarrow \\ \mathcal{M} \end{array} \qquad Z : \Gamma \longrightarrow \mathbb{C} \qquad Z_{\gamma_1+\gamma_2} = Z_{\gamma_1} + Z_{\gamma_2}$$

$\Omega(\gamma)$ piecewise cont. on $\mathcal{M}$.



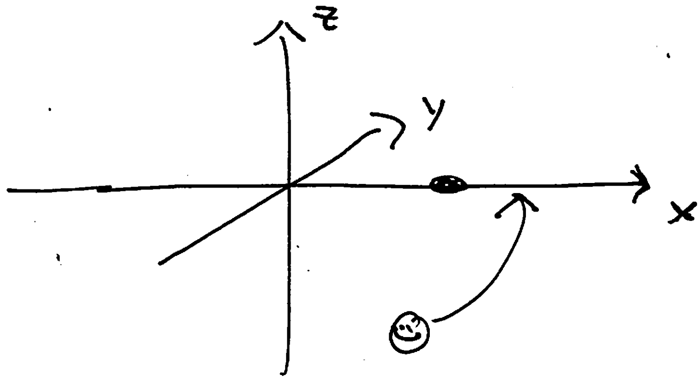
$$\overrightarrow{\prod} K_\gamma^{\Omega_\gamma^{(1)}} = \overrightarrow{\prod} K_\gamma^{\Omega_\gamma^{(2)}}$$

$\mathcal{M}$ = base of Hitchin system on C

$$\Gamma \sim H_1(\Sigma, \mathbb{Z}) \qquad \Sigma \to C \quad (\Sigma \subset T^*C)$$

$$Z_\gamma = \oint_\gamma \lambda$$

GMN (2011) $\longrightarrow$ 2d4d physical context



1+1 QFT

3+1 QFT

$z \in S$ $\quad \mathbb{S}_z$ $\quad$ Vacua $\Longleftrightarrow \pi^{-1}(z) = \{ z^{(i)} \}$

$\gamma_{ij} \subset \Sigma$

(3) R+R CVWCF

A.) LG: $J(X, \alpha)$ $\quad$ X - Kähler
α - closed holo 1-form

Morse theory $\mathcal{X} = \{ \phi : D \longrightarrow X \}$, $D \subset \mathbb{R}$

$\omega = d\lambda$; $\boxed{\alpha = dW ; \; W : X \longrightarrow \mathbb{C}}$ $\rightsquigarrow$ α not closed

$$h = - \int_D [\phi^*(\lambda) - \mathrm{Re}(\zeta^{-1} W) dx]$$

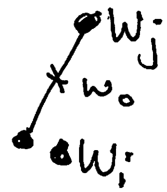
B.) $\delta h = 0$ $\quad \frac{d\phi}{dx} = \begin{cases} \text{grad } \mathrm{Im}(\zeta^{-1} W) \\ \text{Ham } \mathrm{Re}(\zeta^{-1} W) \end{cases}$ $\quad \zeta$-sol eq.

$D = \mathbb{R}$ $\quad \phi_i \longleftarrow$ ——— $\longrightarrow \phi_j$ $\quad \alpha(\phi_i) = 0$

$\underline{i=j}$ $\quad \phi(x)=\phi_i$ **WILL CHANGE**

$i\neq j \quad \mathcal{L}_i(\zeta)=\{\phi_0\in X \mid \phi_0 \underset{x\to-\infty}{\rightsquigarrow} \phi_i\}$

$W: \mathcal{L}_i(\zeta) \longrightarrow$



$CS_{ij} = \mathcal{L}_i(\zeta)\cap\mathcal{L}_j(-\zeta)\cap W^{-1}(w_0) \qquad \underline{\underline{\zeta=\zeta_{ij}}}$

MSW $\quad R_{ij}=\bigoplus_{p\in CS_{ij}} \Psi(p)\,\mathbb{Z}; \quad F=\eta(\not{D}_{\phi_{ij}})$

diff'l: Counting ζ-instantons.

$R_{ii}=\mathbb{Z}$ deg 0. **WILL CHANGE**

$\mu_{ij}\sim \mathrm{Tr}_{R_{ij}}(e^{i\pi F}) \sim \# \,\mathcal{L}_i(e^{-i\epsilon}\zeta_{ij})\cap\mathcal{L}(e^{i\epsilon}\zeta_{ij})$

$Z_{ij} = \int_{\gamma_{ij}} \zeta^{-1}\lambda = \zeta^{-1}(W_i-W_j)$

9:09

Matrices of Complexes $\quad i\neq j \quad \mathcal{E}^W_{ij} = \mathbb{Z}\cdot \mathbb{1}_N + R_{ij}e_{ij}$

$$\mathcal{E}(\text{[web]}) := \bigotimes_{z_{ij}\in \text{[fan]}} \mathcal{E}^W_{ij} := \bigoplus_{i,j}\hat R_{ij}\, e_{ij}$$

$$(\mathcal{E}_1 \otimes \mathcal{E}_2)_{ab} = \bigoplus_c (\mathcal{E}_1)_{ac} \otimes (\mathcal{E}_2)_{cb}$$

$$d = d_1 \otimes 1 + 1 \otimes d_2 + \underbrace{\zeta\text{-instantons}}_{\text{GMW.}}$$

C.) WCF $\quad W(\phi; z); \; z \in \mathcal{M}$

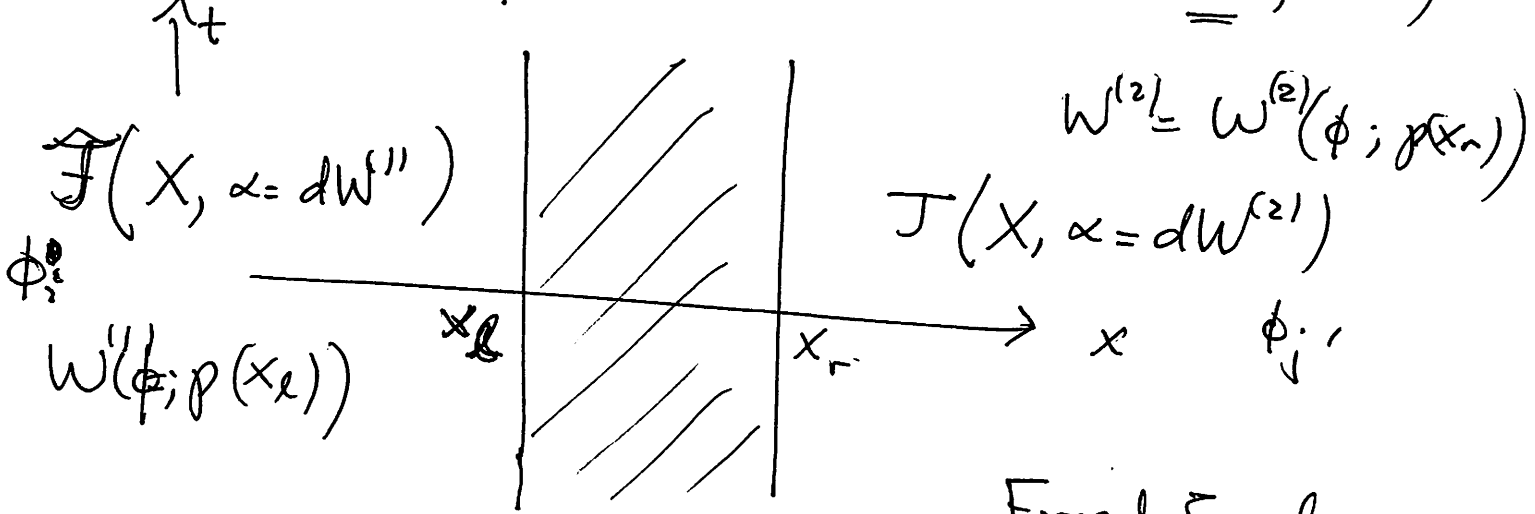
$$\wp : [x_\ell, x_r] \longrightarrow \mathcal{M}$$

$$S_{ij}(x) = \mathbb{1}_N + \mu_{ij}(x) e_{ij}$$

piecewise const. jump across $MSW(ijk) = \{ z \mid Z_{ij} \| Z_{jk} \}$

D.) <u>Interfaces</u>: $\quad$ Morse theory on $\mathcal{H}$

$$h = -\int_D \left(\phi^*(\lambda) - \mathrm{Re}\, \zeta^{-1} W(\phi; \underline{\underline{z(x)}})\, dx \right)$$



Forced ζ-sol eq.

$$\delta h = 0 \qquad \frac{d\phi^I}{dx} = i \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}} (\phi(x); z(x))$$

$$\mathcal{E}[\rho] = \sum_{i,j,j'} \underbrace{R[\rho]_{ij'}}_{\text{MSW.}} e_{ij'}$$

<u>Sol's ?</u>

* Hovering solutions. For fixed x $W(\phi; z(x))$

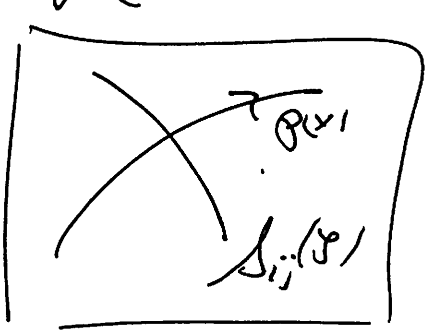
$$\phi(x) = \phi_{i,x} \quad \text{appxt. for } z(x) \text{ slowly varying}$$

$\phi_i^{(1)}$ ———————— $\phi_i^{(2)}$

* <u>S-walls</u> <u>Fix ζ</u>

$$\mathcal{S}_{ij}(\zeta) = \{ z \in \mathcal{M} \mid \exists\ i,j \text{ sol. of phase } \zeta \quad \zeta^{-1}(W_{i,z} - W_{j,z}) \in i\mathbb{R}_+ \}$$

$\mathcal{M}$



$$\rho^s(x) = \begin{cases} \rho(x) & x \le s \\ \rho(s) & x \ge s \end{cases}$$

$$z(s_*) \in \mathcal{S}_{ij'}(\zeta)$$

$\phi_{i,x}^{(1)}$ $\qquad$ $\phi_{ij'}(x, z(s_*))$ $\qquad$ $\phi_{j',x}^{(2)}$

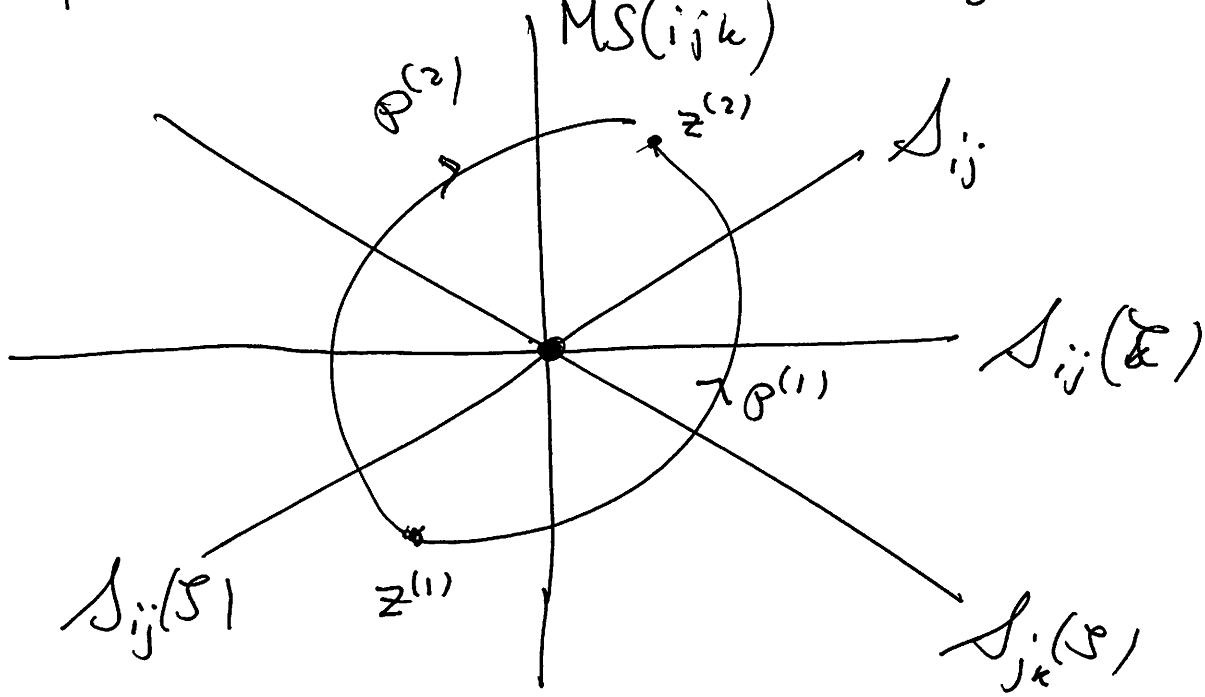
s_*

$$\mathcal{E} \sim \begin{pmatrix} \mathbb{Z} & \mathbb{Z}\cdot\psi \\ 0 & \mathbb{Z} \end{pmatrix}$$

α. $\mathcal{E}[\wp]$ only depends on ~~[crossed out]~~ up to q.i.

β. $$\mathcal{E}(\wp^{s_*+\epsilon}) \underset{q.i.}{\approx} \mathcal{E}(\wp^{s_*-\epsilon})\, \mathcal{E}_{ij}^{W(\phi; z(s_*))}$$

$\alpha, \beta \Longrightarrow$ CVWCF as a corollary



$\wp^{(1)} \sim \wp^{(2)}$ in $\mathcal{M}$

$$\left(\mathcal{E}_{ij}^{w} \mathcal{E}_{jk}^{w} \mathcal{E}_{jk}^{w}\right)^{(L)} \approx \left(\mathcal{E}_{jk}^{w} \mathcal{E}_{ik}^{w} \mathcal{E}_{ij}^{w}\right)^{(R)}$$

9:24

(4) BRANIFY

$$\mathcal{H} \subset \mathbb{R}^2 \Rightarrow \mathfrak{Br}(\mathcal{H}) \quad A_\infty\text{-cat.}$$

$$K(\mathfrak{Br}(\mathcal{H})) \twoheadrightarrow H_n(X, \mathrm{Re}(\bar{\zeta} W) \to \infty)$$

$[\mathcal{L}_i(\zeta)]$ basis.

cross $S_{ij}(\zeta)$
$$[\mathcal{L}_i] \longrightarrow [\mathcal{L}_i] + \mu_{ij}[\mathcal{L}_j]$$
$$[\mathcal{L}_k] \longrightarrow [\mathcal{L}_k] \quad k \neq i$$

Promote $\mathcal{L}_i \Rightarrow \mathbb{L}_i \in \mathrm{Obj}(\mathfrak{Br}(\mathcal{H}))$

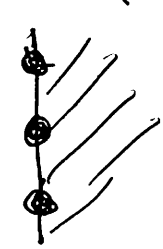
↳ brane wrapping $\mathcal{L}_i$ w/ ~~trivial~~ trivial rk=1 C.P. bundle + con

GMW: $\mathrm{Hop}(\mathbb{L}_i, \mathbb{L}_j) = \hat{R}_{ij}(\mathcal{H})$

$$\mathrm{Ob}(\mathfrak{Br}(\mathcal{H})) = \left\{ \bigoplus \mathcal{E}_i \otimes \mathbb{L}_i \right\}$$

↑ cplxes

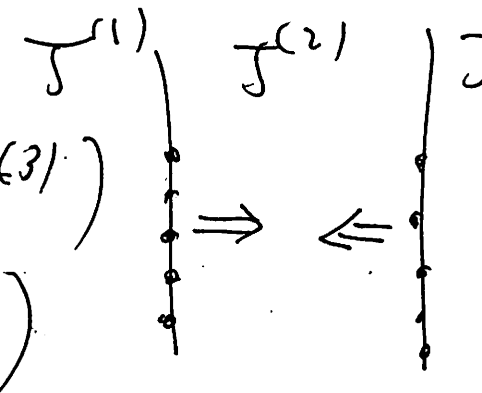
$\exists\, A_\infty$-mult. :



● ≅ Fukaya-Seidel category

Better: Interfaces

$$\mathfrak{Br}(\mathfrak{J}^{(1)},\mathfrak{J}^{(2)}) \times \mathfrak{Br}(\mathfrak{J}^{(2)},\mathfrak{J}^{(3)}) \xrightarrow{\boxtimes} \mathfrak{Br}(\mathfrak{J}^{(1)},\mathfrak{J}^{(3)})$$



$\boxtimes$ ~ ---- -

LG. $\wp : [x_\ell, x_r] \to \mathcal{M} \Rightarrow \mathfrak{I}[\wp] \in$

$$\mathfrak{Br}\big(\mathfrak{J}(X, dW_{x_\ell}), \mathfrak{J}(X, dW_{x_r})\big)$$

[α'] $\mathfrak{I}[\wp]$ only a function [℘].

[β'] ∃ "S-wall interface" $\mathfrak{S}_{ij}^{\pm}$

$$\mathfrak{I}(\wp^{s_* \pm \epsilon}) \simeq \mathfrak{I}(\wp^{s_* \mp \epsilon}) \boxtimes \mathfrak{S}_{ij}^{\pm}$$

* $\mathcal{E}(\mathfrak{I}[\wp]) = \mathcal{E}[\wp]$ Matrix MSW complexes for forced ζ-sol eq

* $\mathcal{E}[\mathfrak{S}_{ij}^{+}] = \mathbb{Z} \cdot 1_N + R_{ij}^{W(\phi, z(s_*))} e_{ij}$

$$(\mathfrak{S}_{ij}^{\pm} \boxtimes \mathfrak{S}_{ik} \boxtimes \mathfrak{S}_{jk})^{(L)} \simeq (\cdots)^{(R)}$$

$$\mathcal{B}r(\phi, \mathcal{T}) = \mathcal{B}r(\mathcal{T})$$

$$\mathcal{H} \longrightarrow \mathcal{H} \boxtimes \mathcal{T}$$

As functor $\mathcal{B}r(\mathcal{T}^{(1)}) \longrightarrow \mathcal{B}r(\mathcal{T}^{(2)})$

- $\boxtimes \mathcal{T}_{ij}$ = categorical mutation of exceptional collections.

CVQCF $\longrightarrow$ categorified braid relation.

9:34

(5) Multi-Valued W

$\alpha \neq dW$ Morse $\longrightarrow$ Morse-Novikov

$$\delta h = -\int_D \phi^* \left(\delta\lambda - \mathrm{Re}\ \zeta^{-1}\delta\alpha\right)$$

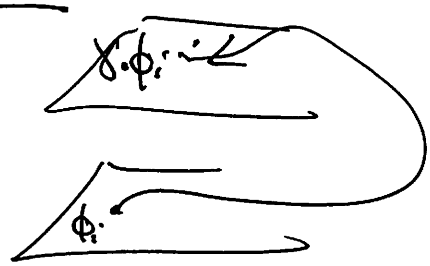
~~[illegible]~~ Cover $\pi : \tilde{X} \longrightarrow X$ $\pi^*\alpha = d\tilde{W}$ (s.v.)

$$J(X,\alpha) \cong \underbrace{J_\Gamma(\tilde{X}, d\tilde{W})}_{\Gamma\text{- equiv.}}$$

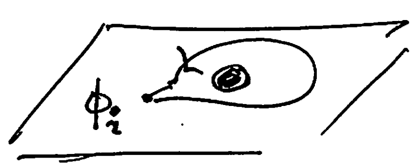
$H_n(\tilde{X}, \mathrm{Re}(\bar{\zeta}\tilde{W}) \to \infty)$ $\mathbb{C}(\Gamma)$ module.

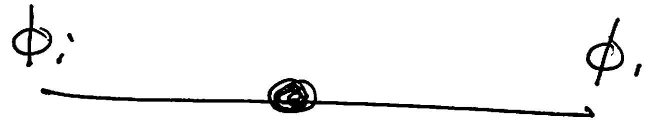
NF1 $\exists$ nontrivial periodic solitons

$\tilde{X}$



X





$$Z(\phi_i^{\mathrm{per}}) = \int_\gamma \alpha$$

$\Longrightarrow$ New walls: $Z(\phi_i^{\mathrm{per}}) \parallel \zeta_K$ "K-walls"

NF2 $\hat{R}_{ii} = \mathrm{Hop}(\mathbb{L}_i, \mathbb{L}_i)$

Now bigraded $(\mathbb{Z}\times\Gamma)$ **FOCK SPACE**

e.g. $\Gamma = \mathbb{Z}$

$$A_{ij}(k,\zeta) = T^k \mathcal{L}_i(\zeta) \cap \mathcal{L}_j(\zeta e^{i\epsilon})$$

$$A_{ij}(q,\zeta) = \sum_{k\in\mathbb{Z}} A_{ij}(k) \cdot q^{f_{ij}+k}$$

$A_{ii}(q,\zeta)$ = Fock space character

$$A_{ii}(q,\zeta_k^+) = \frac{1}{A_{ii}(q^{-1},\zeta_k^{-+})}$$

~~Rescale~~ $\mathcal{L}_i \longrightarrow \dfrac{1}{A_{ii}(T,\zeta_k)} \mathcal{L}_i$

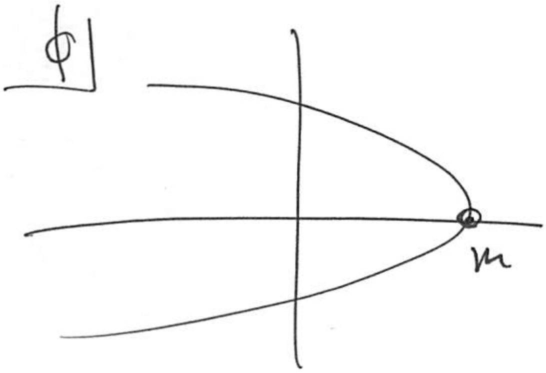
Example $X = \mathbb{C}^*$ $\quad \alpha = \left(\frac{m}{\phi} - 1\right) d\phi$

$\tilde{X} = \mathbb{C}$, $\Gamma = \mathbb{Z}$ $\quad \phi = e^{Y}$

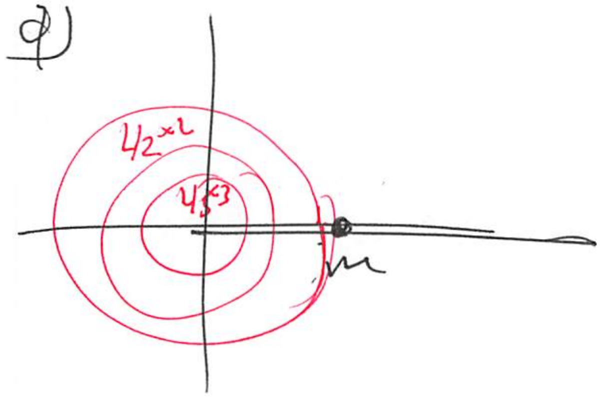
$\phi_{cr} = m \qquad Y^k_{cr} = \log m + 2\pi i k \quad k\in\mathbb{Z}$

$$\zeta_K = i\frac{m}{|m|} = i \quad m>0$$

Thimbles

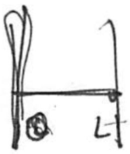


$\zeta = +1$



$\zeta = -1$

$\mathcal{H}(\mathbb{L},\mathbb{L})$



$\mathcal{H}(\mathbb{L},\mathbb{L}) = \bigoplus_{k \le 0} \mathbb{Z}[0,k]$

$\mathbb{Z}[0,0] \oplus \mathbb{Z}[1,1]$

Clifford module

w/ A.K. $\mathbb{L}\left(e^{\frac{i\pi}{2} \pm i\varepsilon}\right) \approx \mathbb{L}\left(e^{\frac{i\pi}{2} \mp i\varepsilon}\right) \boxtimes \mathbb{K}_{\pm}$

$\mathrm{Hop}(\mathbb{K}_{\mp}, \mathbb{K}_{+}) = S(x) \otimes \Lambda(\theta) \otimes \Lambda(\partial_\theta) \otimes S(\partial_x)$

Int. amplitude $\theta\, \partial_x$ Koszul duality

9:44

Exple is universal: $\check{X} = \mathbb{C}$ w/ "twisted mass"

equiv. wrt $U(1)$ on $\mathbb{C}$

$$\text{Action} \longrightarrow \text{Action} + \int |m|^2 \, \|V\|^2$$

V - holo isom V has isolated zeroes

<u>2d4d</u> wcf.

$$SKS \sim SKS$$

$$SS \sim \prod_{1}^{\infty} S \; K \prod_{1}^{\infty} S.$$

at $H_n(--)$ as $\mathbb{Z}[\Gamma]$ module ✓

$$\boxtimes_{1}^{\infty} \sigma_{ij}$$

$\mathbb{CP}^1$ w/ twisted mass

[9:46]

(6) CLASS-S

$$\begin{array}{ccc} T^*C & & \\ \cup & \searrow & \\ \Sigma & \xrightarrow{\pi} & C \ni z \Rightarrow \mathbb{S}_z \end{array}$$

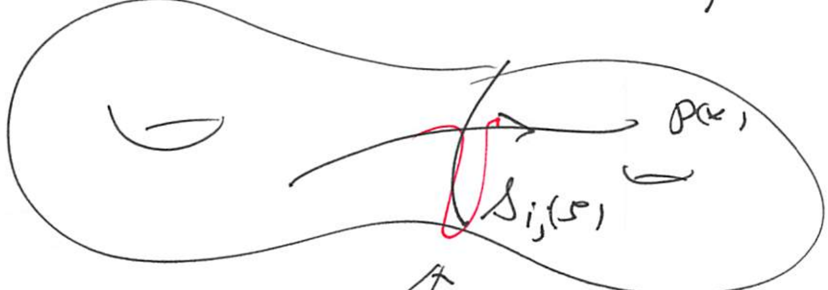
Vacua $\pi^{-1}(z) = \{z^{(i)}\}$

sol. charges $\{ \gamma_{ij} \mid \partial \gamma_{ij} = z^{(i)} - z^{(j)} \}$

$$Z(\gamma_{ij}) = \int_{\gamma_{ij}} \lambda$$

$$\mathcal{S}_{ij}(\varsigma) = \{ z \in C \mid Z(\gamma_{ij}) \in \varsigma \, \mathbb{R}_+ \}$$

$$\mu(\gamma_{ij}) \neq 0$$



↑ Now solns of ς-soli eq.

Kontsevich : $\exists$ 2D LG interp. 22:36

$$L_0 = T^*_z C \qquad L_1 = \Sigma$$

$$\mathcal{H} = \{ \phi : [0,1] \longrightarrow T^*C \mid \phi(0) \in L_0 , \phi(1) \in L_1 \}$$

$$\delta W = \int_0^1 ds \, \omega^{2,0}_{ab} \frac{\partial \varphi^a}{\partial s} \delta \varphi^b$$